**Lecture Notes**

**on**

# Theory of Machine

**(Th.1)**

## $4^{th}$ Semester, Mechanical Engg.

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# THEORY OF MACHINES

**Machine:** A machine is a device which receives energy in some available form and utilizes/transforms it to do some particular type of work.

or

It is the assembly of resistant bodies or links whose relative motions are successfully constrained so that available energy can be converted into useful work.

Example: Lathe, Shaper, Scooter etc.

**Structure:** It is an assemblage of a number of resistant bodies (known as members) having no relative motion between them and meant for carrying loads having straining action.

Example: A railway bridge, a roof truss, machine frames etc.

**Difference between a Machine and a Structure:**

| Machine | Structure |
|---|---|
| • There is relative motion between its members.<br>• It converts available energy into useful work.<br>• Members are meant to transmit motion and forces.<br>• Example : car, lathe etc.. | • There is no relative motion between them.<br>• It does not convert the available energy into useful work.<br>• Members are meant to take up loads only.<br>• Example : bridge, frames, buildings etc.. |



**Theory of Machines:** It may be defined as that branch of Engineering-science, which deals with the study of relative motion between the various parts of a machine, and forces which act on them.

**Branches of Theory of Machines:** It may be sub-divided into the following four branches:

a) ***Kinematics:*** It deals with the relative motion between the various parts of the machines.
b) ***Dynamics:*** It deals with the forces and their effects, while acting upon the machine parts in motion.
c) ***Kinetics:*** It deals with the inertia forces which arise from the combined effect of the mass and motion of the machine parts.
d) ***Statics:*** It deals with the forces and their effects while the machine parts are at rest. The mass of the parts is assumed to be negligible.

**Fundamental Units:** Every quantity is measured in terms of some arbitrary, but internationally accepted units, called fundamental units.

1. Length (L or l), 2. Mass (M or m), and 3. Time (t).

**Derived Units:** Some units are expressed in terms of fundamental units known as derived units, e.g., the units of area, velocity, acceleration, pressure, etc.

**Systems of Units:** It may be defined as the growth of ideas through observation and experimentation.

1. C.G.S. units, 2. F.P.S. units, 3. M.K.S. units and 4. S.I. units.

| | |
|---|---|
| Density (mass density) | $kg/m^3$ |
| Force | N (Newton) |
| Pressure | Pa (Pascal) or $N/m^2$ ( 1 Pa = 1 $N/m^2$) |
| Work, energy (in Joules) | 1 J = 1 N-m |
| Power (in watts) | 1 W = 1 J/s |
| Absolute viscosity | kg/m-s |
| Kinematic viscosity | $m^2/s$ |
| Velocity | m/s |
| Acceleration | $m/s^2$ |
| Angular acceleration | $rad/s^2$ |
| Frequency (in Hertz) | Hz |

**Force:** It may be defined as an agent, which produces or tends to produce, destroy or tends to destroy motion.

**Resultant Force:** If a number of forces are acting simultaneously on a particle, then a single force, which will produce the same effect as that of all the given forces, is known as a resultant force.

a) ***Parallelogram law of forces:*** It states, "If two forces acting simultaneously on a particle be represented in magnitude and direction by the two adjacent sides of a parallelogram taken in order, their resultant may be represented in magnitude and direction by the diagonal of the parallelogram passing through the point."

b) ***Triangle law of forces:*** It states, "If two forces acting simultaneously on a particle be represented in magnitude and direction by the two sides of a triangle taken in order, their resultant may be represented in magnitude and direction by the third side of the triangle taken in opposite order."

c) ***Polygon law of forces:*** It states, "If a number of forces acting simultaneously on a particle be represented in magnitude and direction by the sides of a polygon taken in order, their resultant may be represented in magnitude and direction by the closing side of the polygon taken in opposite order."

**Scalars:** Scalar quantities are those quantities, which have magnitude only, e.g. mass, time, volume, density etc.

**Vectors:** Vector quantities are those quantities which have magnitude as well as direction e.g. velocity, acceleration, force etc.

# Chapter-01: Simple Mechanism

## 1.1 Link, Kinematic Chain, Mechanism, Machine

**Link or Kinematic Link or Element:** Each part of a machine, which moves relative to some other part, is known as a kinematic link (or simply link) or element.

For example, in a reciprocating steam engine

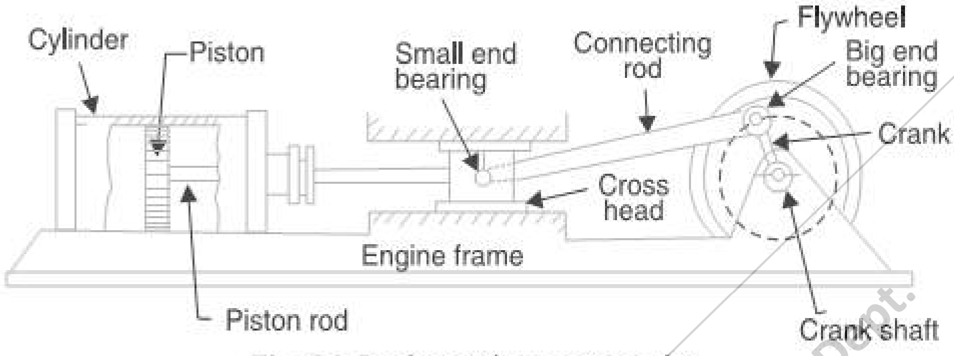



Fig. 5.1. Reciprocating steam engine.

Link-1: Piston, Piston Rod and Cross Head

Link-2: Connecting Rod with Big And Small End Bearings

Link-3: Crank, Crankshaft and Flywheel

Link-4: Cylinder, Engine Frame and Main Bearings

A link or element needs not to be a rigid body, but it must be a resistant body. A body is said to be a resistant body if it is capable of transmitting the required forces with negligible deformation.

Thus a link should have the following two characteristics:

**1.** It should have relative motion, and

**2.** It must be a resistant body.

**Types of Links:**

**a)** ***Rigid link:*** A rigid link is one which does not undergo any deformation while transmitting motion.
Example: Deformation of a connecting rod, crank etc.. is not appreciable., they can be considered as rigid links.

**b)** ***Flexible link:*** A flexible link is one which is partly deformed in a manner not to affect the transmission of motion.
Example: Belts, ropes, chains and wires are flexible links and transmit tensile forces only.

**c)** ***Fluid link:*** A fluid link is one which is formed by having a fluid in a receptacle and the motion is transmitted through the fluid by pressure or compression only.
Example: As in the case of hydraulic presses, jacks and brakes.

**Kinematic Pair:** The two links or elements of a machine, when in contact with each other, are said to form a pair. If the relative motion between them is completely or successfully constrained (i.e. in a definite direction), the pair is known as kinematic pair.

**Types of Constrained Motions:**

1) ***Completely constrained motion:*** When the motion between a pair is limited to a definite direction irrespective of the direction of force applied, then the motion is said to be a completely constrained motion.
   Example: The piston and cylinder (in a steam engine) form a pair and the motion of the piston is limited to a definite direction (i.e. it will only reciprocate) relative to the cylinder irrespective of the direction of motion of the crank.
   The motion of a square bar in a square hole as shown in Fig. 5.2.

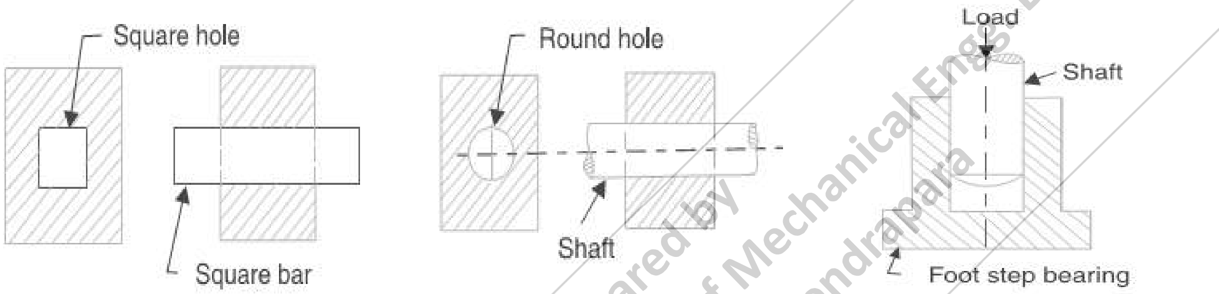



Fig. 5.2. Square bar in a square hole. Fig. 5.4. Shaft in a circular hole. Fig. 5.5. Shaft in a foot step bearing.

2) ***Incompletely constrained motion:*** When the motion between a pair can take place in more than one direction, then the motion is called an incompletely constrained motion.
   Example: A circular bar or shaft in a circular hole, as shown in Fig. 5.4, is an example of an incompletely constrained motion as it may either rotate or slide in a hole. These both motions have no relationship with the other.
3) ***Successfully constrained motion:*** When the motion between the elements, forming a pair, is such that the constrained motion is not completed by itself, but by some other means, then the motion is said to be successfully constrained motion.
   Example: Consider a shaft in a foot-step bearing as shown in Fig. 5.5. The shaft may rotate in a bearing or it may move upwards. This is a case of incompletely constrained motion. But if the load is placed on the shaft to prevent axial upward movement of the shaft, then the motion of the pair is said to be successfully constrained motion.
   The motion of an I.C. engine valve (these are kept on their seat by a spring) and the piston reciprocating inside an engine cylinder are also the examples of successfully constrained motion.



**Classification of Kinematic Pairs:**

**Kinematic pairs**

- Relative motion between the element
  - (i) sliding pair
  - (ii) turning pair
  - (iii) rolling pair
  - (iv) screw pair
  - (v) spherical pair
- Contact between the element
  - (i) lower pair
  - (ii) higher pair
- Types of closure
  - (i) self closed pair
  - (ii) force-closed pair

**1. According to the type of relative motion between the elements:**

***(a) Sliding pair:*** When the two elements of a pair are connected in such a way that one can only slide relative to the other, the pair is known as a sliding pair. A sliding pair has a completely constrained motion.

Ex: The piston and cylinder, cross-head and guides of a reciprocating steam engine.

***(b) Turning pair:*** When the two elements of a pair are connected in such a way that one can only turn or revolve about a fixed axis of another link, the pair is known as turning pair. A turning pair also has a completely constrained motion.

Ex: A shaft with collars at both ends fitted into a circular hole, the crankshaft in a journal bearing in an engine, cycle wheels turning over their axles etc.

***(c) Rolling pair:*** When the two elements of a pair are connected in such a way that one rolls over another fixed link, the pair is known as rolling pair.

Ex: Ball and roller bearings are examples of rolling pair.

***(d) Screw pair:*** When the two elements of a pair are connected in such a way that one element can turn about the other by screw threads, the pair is known as screw pair.

Ex: The lead screw of a lathe with nut, and bolt with a nut.

***(e) Spherical pair:*** When the two elements of a pair are connected in such a way that one element (with spherical shape) turns or swivels about the other fixed element, the pair formed is called a spherical pair.

Ex: The ball and socket joint, attachment of a car mirror, pen stand etc.

**2. According to the type of contact between the elements:**

***(a) Lower pair:*** When the two elements of a pair have a surface contact when relative motion takes place and the surface of one element slides over the surface of the other, the pair formed is known as lower pair.

Ex: It will be seen that sliding pairs, turning pairs and screw pairs form lower pairs.

***(b) Higher pair:*** When the two elements of a pair have a line or point contact when relative motion takes place and the motion between the two elements is partly turning and partly sliding, then the pair is known as higher pair.

Ex: A pair of friction discs, toothed gearing, belt and rope drives, ball and roller bearings and cam and follower are the examples of higher pairs.

**3. According to the type of closure between the elements:**

***(a) Self closed pair*:** When the two elements of a pair are connected together mechanically in such a way that only required kind of relative motion occurs, it is then known as self closed pair.

Ex: The lower pairs are self closed pair.

***(b) Force closed pair:*** When the two elements of a pair are not connected mechanically but are kept in contact by the action of external forces, the pair is said to be a force-closed pair.

Ex: The cam and follower is an example of force closed pair, as it is kept in contact by the forces exerted by spring and gravity.

**Kinematic Chain:** When the kinematic pairs are coupled in such a way that the last link is joined to the first link to transmit definite motion (i.e. completely or successfully constrained motion), it is called a kinematic chain.

If each link is assumed to form two pairs with two adjacent links, then the relations are:

$$l = 2p - 4 \qquad \ldots(i)$$

$$j = \frac{3}{2} l - 2 \qquad \ldots(ii)$$

Where *(p)*: number of pairs, *(l)*: number of links and *(j):* number of joints.

The equations ***(i)*** and ***(ii)*** are applicable only to kinematic chains, in which lower pairs are used. These equations may also be applied to kinematic chains, in which higher pairs are used. In that case each higher pair may be taken as equivalent to two lower pairs with an additional element or link.

**Example 1:** Consider the arrangement of three links AB, BC and CA with pin joints at A , B and C.

Number of links, $l = 3$

Number of pairs, $p = 3$

and number of joints, $j = 3$

From equation (*i*), $l = 2p - 4$

or $3 = 2 \times 3 - 4 = 2$

*i.e.* L.H.S. > R.H.S.

Now from equation (*ii*),

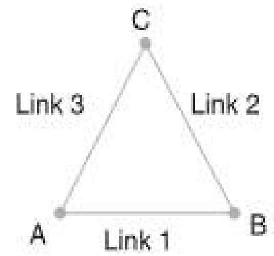



**Fig. 5.6.** Arrangement of three links.

$$j = \frac{3}{2} l - 2 \quad \text{or} \quad 3 = \frac{3}{2} \times 3 - 2 = 2.5$$

i.e. L.H.S. > R.H.S.

Since the arrangement of three links, as shown in Fig. 5.6, does not satisfy the equations *(i)* and *(ii)* and the left hand side is greater than the right hand side, therefore it is not a kinematic chain and hence no relative motion is possible. Such type of chain is called locked chain and forms a rigid frame or structure which is used in bridges and trusses.

**Example 2:** Consider the arrangement of four links A B, BC, CD and DA.

$l = 4$, $p = 4$, and $j = 4$

From equation (*i*), $l = 2p - 4$

$4 = 2 \times 4 - 4 = 4$

*i.e.* L.H.S. = R.H.S.

From equation (*ii*), $j = \frac{3}{2} l - 2$

$4 = \frac{3}{2} \times 4 - 2 = 4$

*i.e.* L.H.S. = R.H.S.

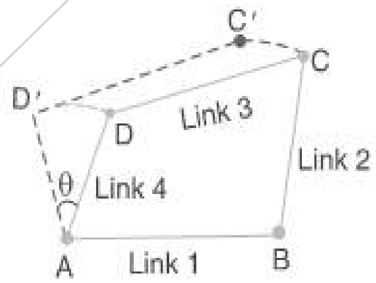



**Fig. 5.7.** Arrangement of four links.

Since the arrangement of four links, satisfy the equations *(i)* and *(ii)*, therefore it is a kinematic chain of one degree of freedom.

A chain in which a single link such as AD is sufficient to define the position of all other links, it is then called a kinematic chain of one degree of freedom.

A little consideration will show that, if a definite displacement (say θ) is given to the link AD, keeping the link AB fixed, then the resulting displacements of the remaining two links BC and CD are also perfectly definite. Thus we see that in a four bar chain, the relative motion is completely constrained. Hence it may be called as a constrained kinematic chain, and it is the basis of all machines.

**Example 3:** Consider the arrangement of five links.

$l = 5, p = 5$, and $j = 5$

From equation (*i*),

$$l = 2p - 4 \quad \text{or} \quad 5 = 2 \times 5 - 4 = 6$$

*i.e.* L.H.S. < R.H.S.

From equation (*ii*),

$$j = \frac{3}{2}l - 2 \quad \text{or} \quad 5 = \frac{3}{2} \times 5 - 2 = 5.5$$

*i.e.* L.H.S. < R.H.S.

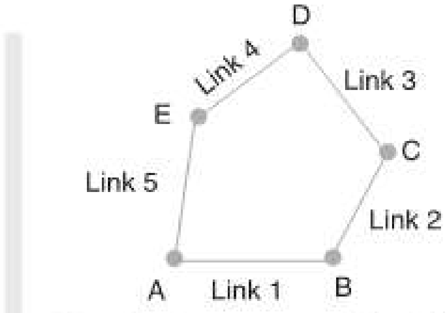


**Fig. 5.8.** Arrangement of five links.

Since the arrangement of five links, as shown in Fig. 5.8 does not satisfy the equations and left hand side is less than right hand side, therefore it is not a kinematic chain. Such a type of chain is called **unconstrained chain** i.e. the relative motion is not completely constrained. This type of chain is of little practical importance.

**Example 4:** Consider the arrangement of six links. This chain is formed by adding two more links in such a way that these two links form a pair with the existing links as well as form themselves a pair. In this case l = 6, p = 5, and j = 7

From equation (*i*),

$$l = 2p - 4 \quad \text{or} \quad 6 = 2 \times 5 - 4 = 6$$

*i.e.* L.H.S. = R.H.S.

From equation (*ii*),

$$j = \frac{3}{2}l - 2 \quad \text{or} \quad 7 = \frac{3}{2} \times 6 - 2 = 7$$

*i.e.* L.H.S. = R.H.S.

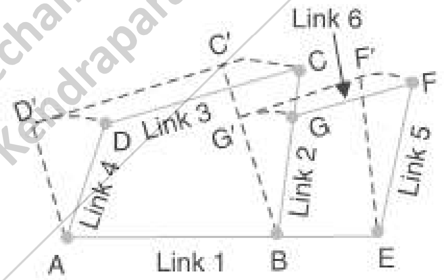


**Fig. 5.9.** Arrangement of six links.

Since the arrangement of six links, as shown in Fig. 5.9, satisfies the equations (*i.e.* left hand side is equal to right hand side), therefore it is a kinematic chain.

**Note :** A chain having more than four links is known as **compound kinematic chain.**



**Mechanism:**

- When one of the links of a kinematic chain is fixed, the chain is known as mechanism.
- It may be used for transmitting or transforming motion e.g. engine indicators, typewriter etc.
- A mechanism with four links is known as ***simple mechanism***, and the mechanism with more than four links is known as ***compound mechanism***.

**Machine:**

- When a mechanism is required to transmit power or to do some particular type of work, it then becomes a machine.
- A mechanism may be regarded as a machine in which each part is reduced to the simplest form to transmit the required motion.

## 1.2 Inversion, four bar link mechanism and its inversion

**Inversion:** The method of obtaining different mechanisms by fixing different links in a kinematic chain, is known as **inversion of the mechanism**.

Suppose number of links of a kinematic chain = L, So L different mechanisms may be obtained by fixing each of the lines in turn. Each mechanism is termed as inversion.

**Types of Kinematic Chains:**

1. Four bar chain or quadric cycle chain
2. Single slider crank chain
3. Double slider crank chain

**1. Four Bar Chain or Quadric Cycle Chain:**

The simplest and the basic kinematic chain is a four bar chain or quadric cycle chain. It consists of four links, each of them forms a turning pair at A, B, C and D. The four links may be of different lengths.

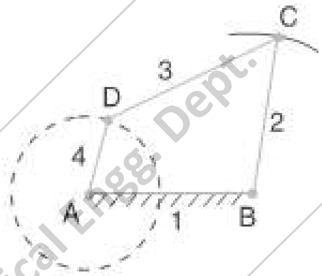


**Fig. 5.18.** Four bar chain.

**Grashof's law:** According to this law for a four bar mechanism, the sum of the shortest and longest link lengths should not be greater than the sum of the remaining two link lengths if there is to be continuous relative motion between the two links.

- In a four bar chain, one of the links, in particular the shortest link, will make a complete revolution relative to the other three links, if it satisfies the Grashof's law. Such a link is known as ***crank*** or ***driver***, AD (link 4) is a ***crank***.
- The link BC (link 2) which makes a partial rotation or oscillates is known as lever or ***rocker*** or ***follower*** and the link CD (link 3) which connects the crank and lever is called ***connecting* rod** or ***coupler***.
- The fixed link AB (link 1) is known as ***frame*** of the mechanism.
- When the crank (link 4) is the ***driver***, the mechanism is transforming rotary motion into oscillating motion.

**Inversions of Four Bar Chain:**

***a) Beam engine (crank and lever mechanism):***

In this mechanism, when the crank rotates about the fixed centre A, the lever oscillates about a fixed centre D. The end E of the lever CDE is connected to a piston rod which reciprocates due to the rotation of the crank. In other words, the purpose of this mechanism is to convert rotary motion into reciprocating motion.

Rotary motion ------> Reciprocating motion --------> To power steam ship

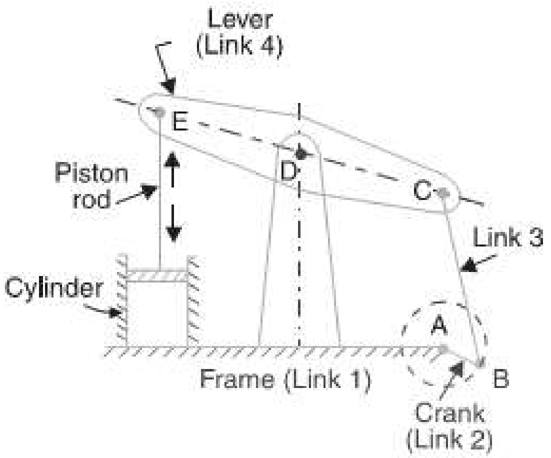



Fig. 5.19. Beam engine.

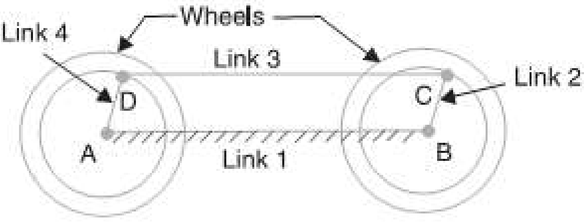



Fig. 5.20. Coupling rod of a locomotive.

***b) Coupling rod of a locomotive (Double crank mechanism):***

- In this mechanism, the links AD and BC (having equal length) act as cranks and are connected to the respective wheels.
- The link CD acts as a coupling rod and the link AB is fixed in order to maintain a constant centre to centre distance between them.
- This mechanism is meant for transmitting rotary motion from one wheel to the other wheel.

***c) Watt's indicator mechanism (Double lever mechanism):***

- The four links are: fixed link at A, link AC, link CE and link BFD. It may be noted that BF and FD form one link because these two parts have no relative motion between them.
- The links CE and BFD act as levers.
- The displacement of the link BFD is directly proportional to the pressure of gas or steam which acts on the indicator plunger.
- On any small displacement of the mechanism, the tracing point E at the end of the link CE traces out approximately a straight line.

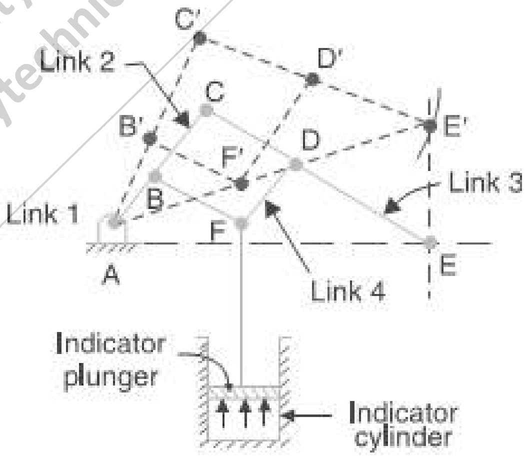



Fig. 5.21. Watt's indicator mechanism.

**2. Single Slider Crank Chain:**

A single slider crank chain is a modification of the basic four bar chain. It consist of one sliding pair and three turning pairs. It is, usually, found in reciprocating steam engine mechanism. This type of mechanism converts rotary motion into reciprocating motion and vice versa.

In a single slider crank chain the links 1 and 2, links 2 and 3, and links 3 and 4 form three turning pairs while the links 4 and 1 form a sliding pair.

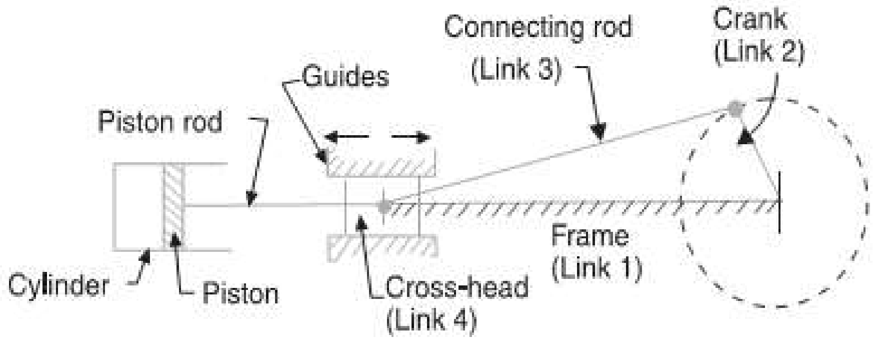



Fig. 5.22. Single slider crank chain.

- The link 1 corresponds to the frame of the engine, which is fixed.
- The link 2 corresponds to the crank;
- link 3 corresponds to the connecting rod and
- link 4 corresponds to cross-head.

As the crank rotates, the cross-head reciprocates in the guides and thus the piston reciprocates in the cylinder.

**Inversions of Single Slider Crank Chain:**

***a) Pendulum pump or Bull engine:***

- In this mechanism, the inversion is obtained by fixing the cylinder or link 4 (i.e. sliding pair).
- In this case, when the crank (link 2) rotates, the connecting rod (link 3) oscillates about a pin pivoted to the fixed link 4 at A and the piston attached to the piston rod (link 1) reciprocates.
- The duplex pump which is used to supply feed water to boilers have two pistons attached to link 1.



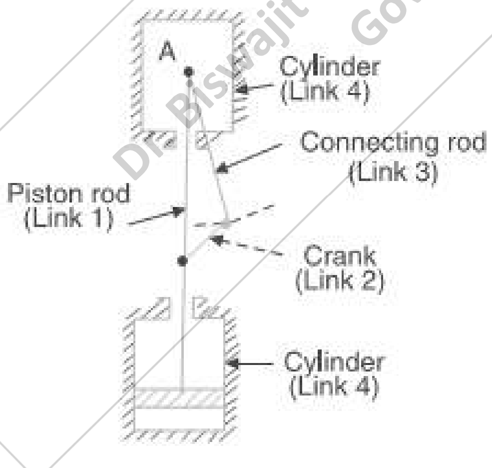



Fig. 5.23. Pendulum pump.

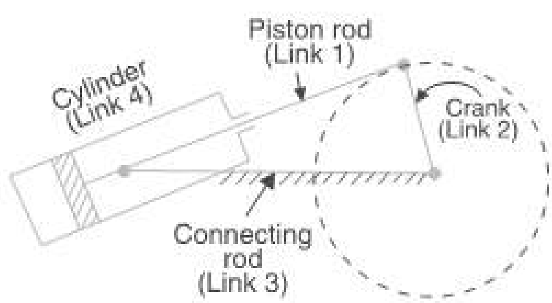



Fig. 5.24. Oscillating cylinder engine.

***b) Oscillating cylinder engine:***

- The oscillating cylinder engine mechanism is used to convert reciprocating motion into rotary motion.

- In this mechanism, the link 3 forming the turning pair is fixed. The link 3 corresponds to the connecting rod of a reciprocating steam engine mechanism.
- When the crank (link 2) rotates, the piston attached to piston rod (link 1) reciprocates and the cylinder (link 4) oscillates about a pin pivoted to the fixed link at A.

***c) Rotary internal combustion engine or Gnome engine:***

- Sometimes back, rotary internal combustion engines were used in aviation. But now-a-days gas turbines are used in its place.

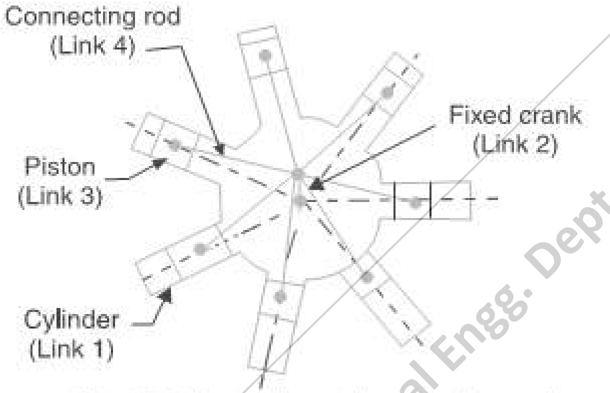
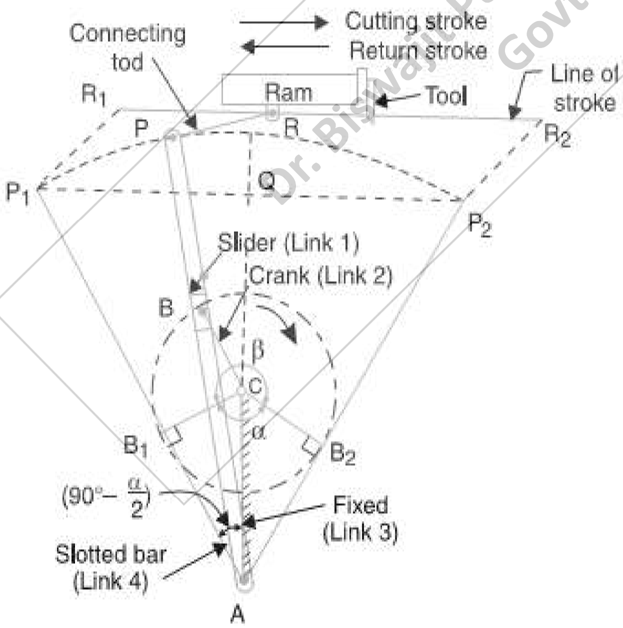




**Fig. 5.25.** Rotary internal combustion engine.

- It consists of seven cylinders in one plane and all revolves about fixed centre D, while the crank (link 2) is fixed.
- In this mechanism, when the connecting rod (link 4) rotates, the piston (link 3) reciprocates inside the cylinders forming link 1.

***d) Crank and slotted lever quick return motion mechanism:***

- This mechanism is mostly used in shaping machines, slotting machines and in rotary internal combustion engines.



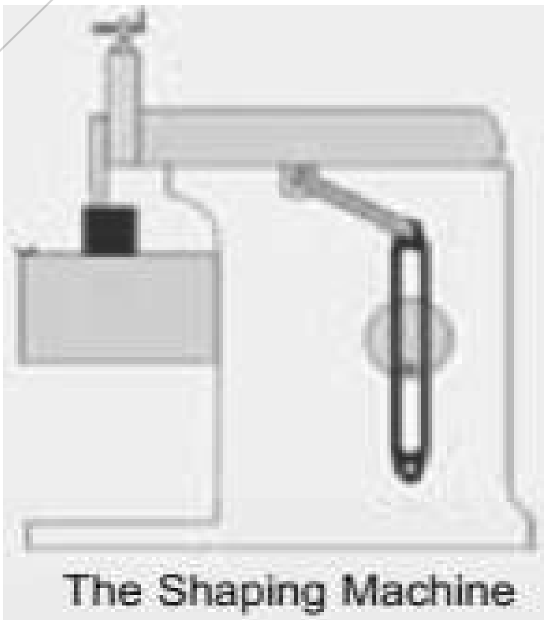



**Fig. 5.26.** Crank and slotted lever quick return motion mechanism.

- In this mechanism, the link AC (i.e. link 3) forming the turning pair is fixed. The link 3 corresponds to the connecting rod of a reciprocating steam engine.
- The driving crank CB revolves with uniform angular speed about the fixed centre C.
- A sliding block attached to the crank pin at B slides along the slotted bar AP and thus causes AP to oscillate about the pivoted point A.
- A short link PR transmits the motion from AP to the ram which carries the tool and reciprocates along the line of stroke $R_1R_2$.
- The line of stroke of the ram (i.e. $R_1R_2$) is perpendicular to AC produced.

In the extreme positions, $AP_1$ and $AP_2$ are tangential to the circle and the cutting tool is at the end of the stroke. The forward or cutting stroke occurs when the crank rotates from the position $CB_1$ to $CB_2$ (or through an angle β) in the clockwise direction. The return stroke occurs when the crank rotates from the position $CB_2$ to $CB_1$ (or through angle α) in the clockwise direction. Since the crank has uniform angular speed, therefore,

$$\frac{\text{Time of cutting stroke}}{\text{Time of return stroke}} = \frac{\beta}{\alpha} = \frac{\beta}{360^\circ - \beta} \text{ or } \frac{360^\circ - \alpha}{\alpha}$$

Since the tool travels a distance of $R_1R_2$ during cutting and return stroke, therefore travel of the tool or length of stroke,

$$= R_1R_2 = P_1P_2 = 2P_1Q = 2AP_1 \sin \angle P_1AQ$$

$$= 2AP_1 \sin\left(90^\circ - \frac{\alpha}{2}\right) = 2AP \cos\frac{\alpha}{2} \qquad \text{..... } (\because AP_1 = AP)$$

$$= 2AP \times \frac{CB_1}{AC} \qquad \cdots \left(\because \cos\frac{\alpha}{2} = \frac{CB_1}{AC}\right)$$

$$= 2AP \times \frac{CB}{AC} \qquad \text{... } (\because CB_1 = CB)$$

**Note:** From Fig. 5.26, we see that the angle β made by the forward or cutting stroke is greater than the angle α described by the return stroke. Since the crank rotates with uniform angular speed, therefore the return stroke is completed within shorter time. Thus it is called quick return motion mechanism.

***e) Whitworth quick return motion mechanism:***

- This mechanism is mostly used in shaping and slotting machines.
- In this mechanism, the link CD (link 2) forming the turning pair is fixed. The link 2 corresponds to a crank in a reciprocating steam engine.
- The driving crank CA (link 3) rotates at a uniform angular speed.
- The slider (link 4) attached to the crank pin at A slides along the slotted bar PA (link 1) which oscillates at a pivoted point D.
- The connecting rod PR carries the ram at R to which a cutting tool is fixed.

- The motion of the tool is constrained along the line RD produced, i.e. along a line passing through D and perpendicular to CD.

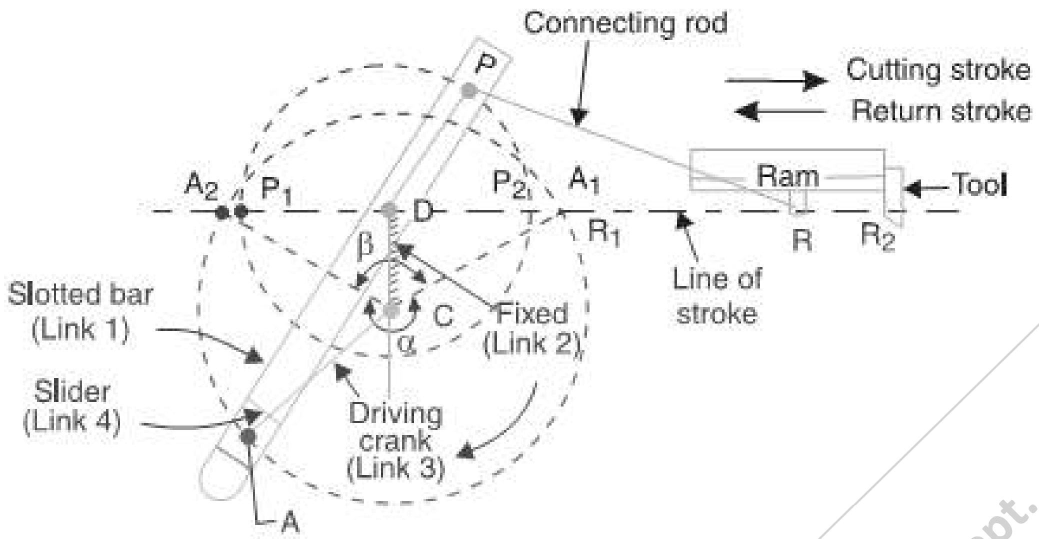



**Fig. 5.27.** Whitworth quick return motion mechanism.

When the driving crank *CA* moves from the position $CA_1$ to $CA_2$ (or the link *DP* from the position $DP_1$ to $DP_2$) through an angle α in the clockwise direction, the tool moves from the left hand end of its stroke to the right hand end through a distance 2 *PD*.

Now when the driving crank moves from the position $CA_2$ to $CA_1$ (or the link *DP* from $DP_2$ to $DP_1$) through an angle *β* in the clockwise direction, the tool moves back from right hand end of its stroke to the left hand end.

A little consideration will show that the time taken during the left to right movement of the ram (i.e. during forward or cutting stroke) will be equal to the time taken by the driving crank to move from $CA_1$ to $CA_2$. Similarly, the time taken during the right to left movement of the ram (or during the idle or return stroke) will be equal to the time taken by the driving crank to move from $CA_2$ to $CA_1$.

Since the crank link *CA* rotates at uniform angular velocity therefore time taken during the cutting stroke (or forward stroke) is more than the time taken during the return stroke. In other words, the mean speed of the ram during cutting stroke is less than the mean speed during the return stroke. The ratio between the time taken during the cutting and return strokes is given by,

$$\frac{\text{Time of cutting stroke}}{\text{Time of return stroke}} = \frac{\alpha}{\beta} = \frac{\alpha}{360^\circ - \alpha} \quad \text{or} \quad \frac{360^\circ - \beta}{\beta}$$

**Note:** In order to find the length of effective stroke $R_1R_2$, mark $P_1R_1 = P_2R_2 = PR$. The length of effective stroke is also equal to 2 *PD*.

**Example 5.1.** *A crank and slotted lever mechanism used in a shaper has a centre distance of 300 mm between the centre of oscillation of the slotted lever and the centre of rotation of the crank. The radius of the crank is 120 mm. Find the ratio of the time of cutting to the time of return stroke.*

**Solution.** Given : $AC = 300$ mm ; $CB_1 = 120$ mm

The extreme positions of the crank are shown in Fig. 5.28. We know that

$$\sin \angle CAB_1 = \sin(90° - \alpha/2)$$

$$= \frac{CB_1}{AC} = \frac{120}{300} = 0.4$$

$$\therefore \quad \angle CAB_1 = 90° - \alpha/2$$

$$= \sin^{-1} 0.4 = 23.6°$$

or $$\alpha/2 = 90° - 23.6° = 66.4°$$

and $$\alpha = 2 \times 66.4 = 132.8°$$

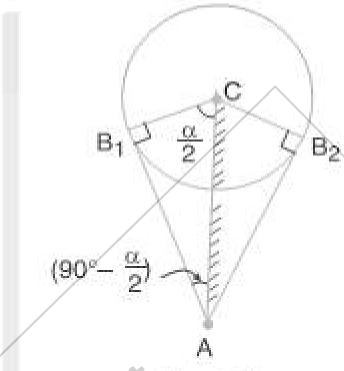



Fig. 5.28

We know that

$$\frac{\text{Time of cutting stroke}}{\text{Time of return stroke}} = \frac{360° - \alpha}{\alpha} = \frac{360° - 132.8°}{132.8°} = 1.72 \text{ Ans.}$$

**Example 5.2.** *In a crank and slotted lever quick return motion mechanism, the distance between the fixed centres is 240 mm and the length of the driving crank is 120 mm. Find the inclination of the slotted bar with the vertical in the extreme position and the time ratio of cutting stroke to the return stroke.*

*If the length of the slotted bar is 450 mm, find the length of the stroke if the line of stroke passes through the extreme positions of the free end of the lever.*

**Solution.** Given : $AC = 240$ mm ; $CB_1 = 120$ mm ; $AP_1 = 450$ mm

***Inclination of the slotted bar with the vertical***

Let $\angle CAB_1$ = Inclination of the slotted bar with the vertical.

The extreme positions of the crank are shown in Fig. 5.29. We know that

$$\sin \angle CAB_1 = \sin\left(90° - \frac{\alpha}{2}\right)$$

$$= \frac{B_1C}{AC} = \frac{120}{240} = 0.5$$

$$\therefore \angle CAB_1 = 90° - \frac{\alpha}{2}$$

$$= \sin^{-1} 0.5 = 30° \text{ Ans.}$$

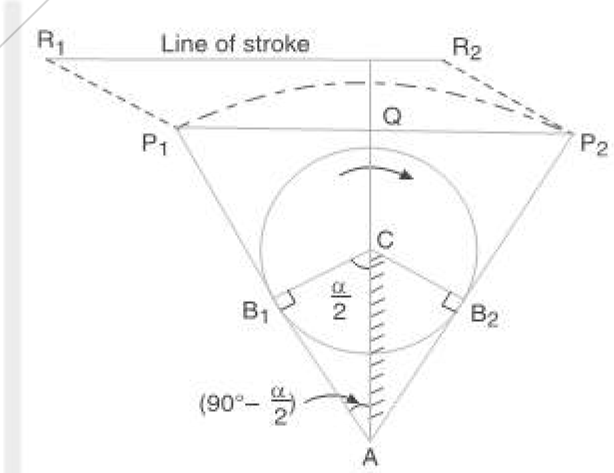



Fig. 5.29

***Time ratio of cutting stroke to the return stroke***

We know that

$$90° - \alpha/2 = 30°$$

$$\therefore \quad \alpha/2 = 90° - 30° = 60°$$

or $$\alpha = 2 \times 60° = 120°$$

$$\therefore \quad \frac{\text{Time of cutting stroke}}{\text{Time of return stroke}} = \frac{360° - \alpha}{\alpha} = \frac{360° - 120°}{120°} = 2 \text{ Ans.}$$

*Length of the stroke*

We know that length of the stroke,

$$R_1R_2 = P_1P_2 = 2\,P_1Q = 2\,AP_1 \sin(90° - \alpha/2)$$
$$= 2 \times 450 \sin(90° - 60°) = 900 \times 0.5 = 450 \text{ mm } \textbf{Ans.}$$

**Example 5.3.** *Fig. 5.30 shows the lay out of a quick return mechanism of the oscillating link type, for a special purpose machine. The driving crank BC is 30 mm long and time ratio of the working stroke to the return stroke is to be 1.7. If the length of the working stroke of R is 120 mm, determine the dimensions of AC and AP.*

**Solution.** Given : $BC = 30$ mm ; $R_1R_2 = 120$ mm ; Time ratio of working stroke to the return stroke = 1.7

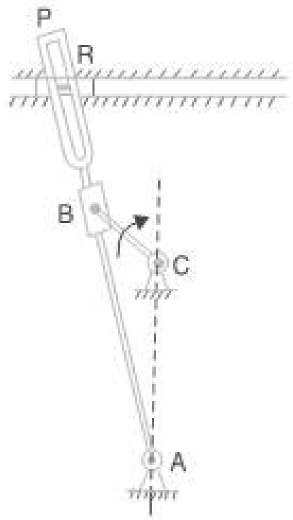



Fig. 5.30

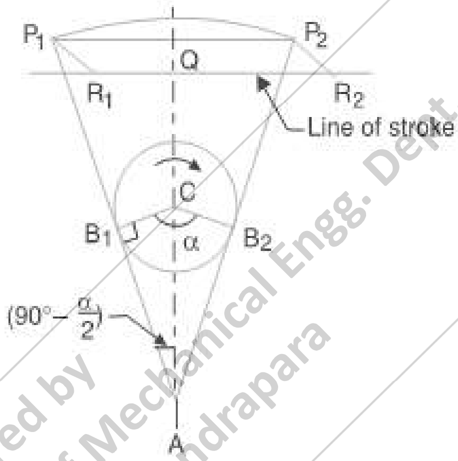



Fig. 5.31

We know that

$$\frac{\text{Time of working stroke}}{\text{Time of return stroke}} = \frac{360 - \alpha}{\alpha} \quad \text{or} \quad 1.7 = \frac{360 - \alpha}{\alpha}$$

$\therefore \qquad \alpha = 133.3° \quad$ **or** $\quad \alpha/2 = 66.65°$

The extreme positions of the crank are shown in Fig. 5.31. From right angled triangle $A\,B_1C$, we find that

$$\sin(90° - \alpha/2) = \frac{B_1C}{AC} \quad \text{or} \quad AC = \frac{B_1C}{\sin(90° - \alpha/2)} = \frac{BC}{\cos\alpha/2}$$

... ($\because B_1C = BC$)

$$\therefore \qquad AC = \frac{30}{\cos 66.65°} = \frac{30}{0.3963} = 75.7 \text{ mm } \textbf{Ans.}$$

We know that length of stroke,

$$R_1R_2 = P_1P_2 = 2P_1Q = 2\,AP_1 \sin(90° - \alpha/2) = 2\,AP_1 \cos\alpha/2$$
$$120 = 2\,AP \cos 66.65° = 0.7926\,AP \qquad \text{... } (\because AP_1 = AP)$$

$$\therefore \qquad AP = 120 / 0.7926 = 151.4 \text{ mm } \textbf{Ans.}$$

**Example 5.4.** *In a Whitworth quick return motion mechanism, as shown in Fig. 5.32, the distance between the fixed centers is 50 mm and the length of the driving crank is 75 mm. The length of the slotted lever is 150 mm and the length of the connecting rod is 135 mm. Find the ratio of the time of cutting stroke to the time of return stroke and also the effective stroke.*

**Solution.** Given : $CD = 50$ mm ; $CA = 75$ mm ; $PA = 150$ mm ; $PR = 135$ mm

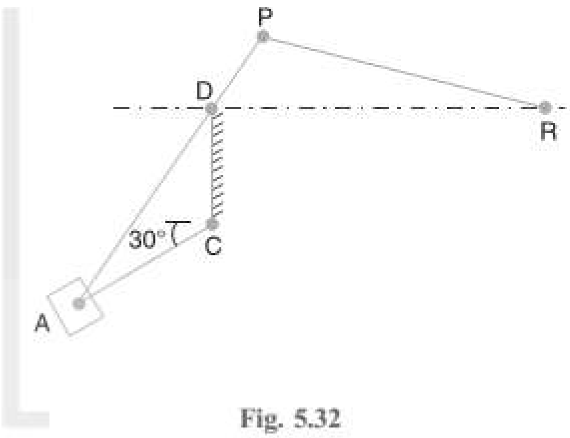



Fig. 5.32

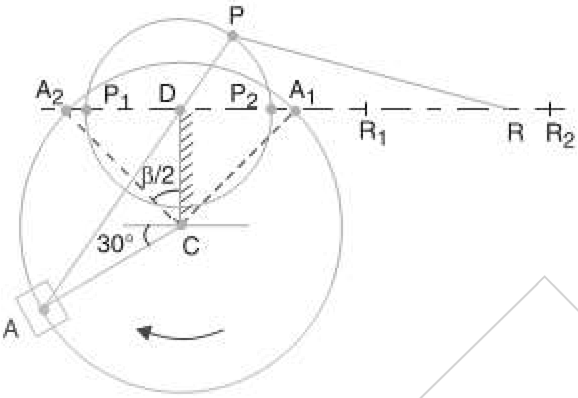



Fig. 5.33

The extreme positions of the driving crank are shown in Fig. 5.33. From the geometry of the figure,

$$\cos\beta/2 = \frac{CD}{CA_2} = \frac{50}{75} = 0.667 \quad \ldots (\because CA_2 = CA)$$

$$\therefore \quad \beta/2 = 48.2° \quad \text{or} \quad \beta = 96.4°$$

***Ratio of the time of cutting stroke to the time of return stroke***

We know that

$$\frac{\text{Time of cutting stroke}}{\text{Time of return stroke}} = \frac{360 - \beta}{\beta} = \frac{360 - 96.4}{96.4} = 2.735 \text{ **Ans.**}$$

***Length of effective stroke***

In order to find the length of effective stroke (*i.e.* $R_1R_2$), draw the space diagram of the mechanism to some suitable scale, as shown in Fig. 5.33. Mark $P_1R_2 = P_2R_2 = PR$. Therefore by measurement we find that,

Length of effective stroke = $R_1R_2$ = 87.5 mm **Ans.**

### 3. Double Slider Crank Chain:

A kinematic chain which consists of two turning pairs and two sliding pairs is known as ***double slider crank chain***, as shown in **Fig. 5.34**. We see that the link 2 and link 1 form one turning pair and link 2 and link 3 form the second turning pair. The link 3 and link 4 form one sliding pair and link 1 and link 4 form the second sliding pair.

**Inversions of Double Slider Crank Chain:**

***a) Elliptical trammels:***

- It is an instrument used for drawing ellipses.
- This inversion is obtained by fixing the slotted plate (link 4), as shown in **Fig. 5.34**. The fixed plate or link 4 has two straight grooves cut in it, at right angles to each other.
- The link 1 and link 3 are known as sliders and form sliding pairs with link 4.
- The link AB (link 2) is a bar which forms turning pair with links 1 and 3.
- When the links 1 and 3 slide along their respective grooves, any point on the link 2 such as P traces out an ellipse on the surface of link 4.

A little consideration will show that AP and BP are the semi-major axis and semi-minor axis of the ellipse respectively. This can be proved as follows.

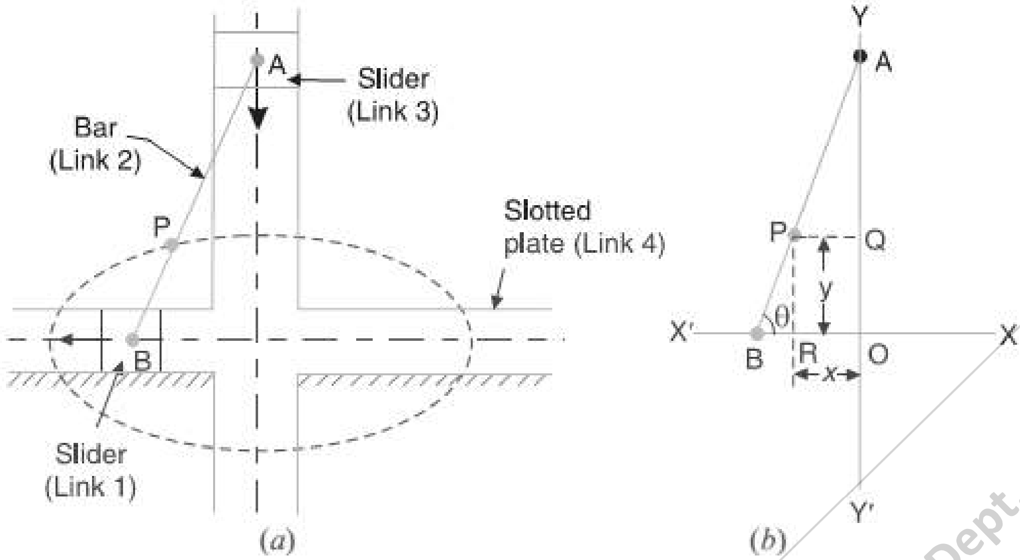



Fig. 5.34. Elliptical trammels.

Let us take *OX* and *OY* as horizontal and vertical axes and let the link *BA* is inclined at an angle *θ* with the horizontal. Now the co-ordinates of the point *P* on the link *BA* will be,

$$x = PQ = AP \cos\theta; \text{ and } y = PR = BP \sin\theta$$

or
$$\frac{x}{AP} = \cos\theta; \text{ and } \frac{y}{BP} = \sin\theta$$

Squaring and adding,

$$\frac{x^2}{(AP)^2} + \frac{y^2}{(BP)^2} = \cos^2\theta + \sin^2\theta = 1$$

This is the equation of an ellipse. Hence the path traced by point *P* is an ellipse whose semi-major axis is *AP* and semi-minor axis is *BP*.

***b) Scotch yoke mechanism:***

- This mechanism is used for converting rotary motion into a reciprocating motion.
- The inversion is obtained by fixing either the link 1 or link 3.
- In **Fig. 5.35**, link 1 is fixed and it guides the frame.

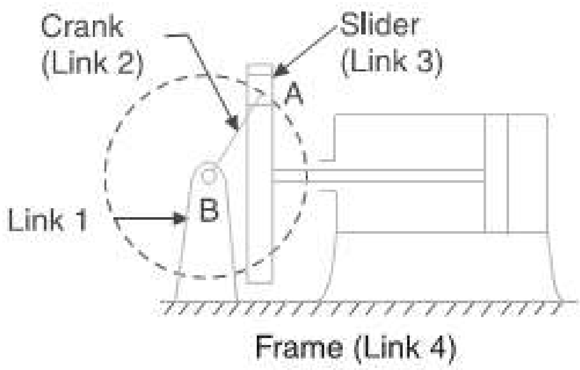



Fig. 5.35. Scotch yoke mechanism.

- In this mechanism, when the link 2 (which corresponds to crank) rotates about B as centre, the link 4 (which corresponds to a frame) reciprocates.

***c) Oldham's coupling:***

- An Oldham's coupling is used for connecting two parallel shafts whose axes are at a small distance apart.
- The shafts are coupled in such a way that if one shaft rotates, the other shaft also rotates at the same speed.
- This inversion is obtained by fixing the link 2, as shown in **Fig. 5.36 (a)**.
- The shafts to be connected have two flanges (link 1 and link 3) rigidly fastened at their ends by forging.
- The link 1 and link 3 form turning pairs with link 2. These flanges have diametrical slots cut in their inner faces, as shown in **Fig. 5.36 (b)**.
- The intermediate piece (link 4) which is a circular disc, have two tongues (i.e. diametrical projections) $T_1$ and $T_2$ on each face at right angles to each other shown in **Fig. 5.36 (c)**.
- The tongues on the link 4 closely fit into the slots in the two flanges (link 1 and link 3). The link 4 can slide or reciprocate in the slots in the flanges.

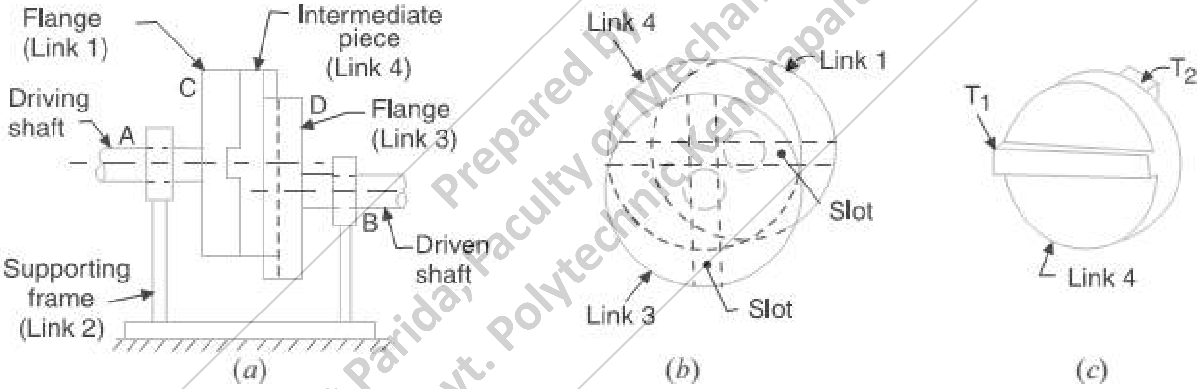



Fig. 5.36. Oldham's coupling.



- When the driving shaft A is rotated, the flange C (link 1) causes the intermediate piece (link 4) to rotate at the same angle through which the flange has rotated, and it further rotates the flange D (link 3) at the same angle and thus the shaft B rotates. Hence links 1, 3 and 4 have the same angular velocity at every instant.
- A little consideration will show, that there is a sliding motion between the link 4 and each of the other links 1 and 3.
- If the distance between the axes of the shafts is constant, the centre of intermediate piece will describe a circle of radius equal to the distance between the axes of the two shafts.
- Therefore, the maximum sliding speed of each tongue along its slot is equal to the peripheral velocity of the centre of the disc along its circular path.

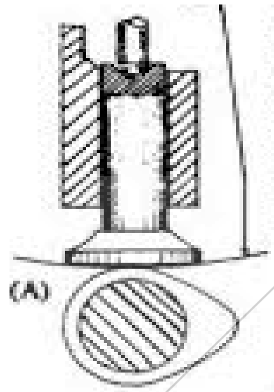
  Let $\omega$ = Angular velocity of each shaft in rad/s, and

  $r$ = Distance between the axes of the shafts in metres.

  $\therefore$ Maximum sliding speed of each tongue (in m/s), $v = \omega . r$

## 1.4 Cam and Followers

- A ***cam*** is a rotating machine element which gives reciprocating or oscillating motion to another element known as ***follower***.
- The cam and the follower have a ***line contact*** and constitute a ***higher pair***.
- A cam is difficult to manufacture especially when it is to be produced in small quantities. When produced on a mass scale, it is manufactured by a punch press, by die casting or by milling from the master cam.




- The cams are usually rotated at uniform speed by a shaft, but the follower motion is predetermined and will be according to the shape of the cam.

***Ex:*** The cams are widely used for operating the inlet and exhaust valves of internal combustion engines, automatic attachment of machineries, paper cutting machines, spinning and weaving textile machineries, feed mechanism of automatic lathes etc.

**Classification of Followers:**

**Follower**

**Surface in contact**
(i) Knife edge follower
(ii) Roller follower
(iii) Flat faced or
(iv) Spherical faced follower

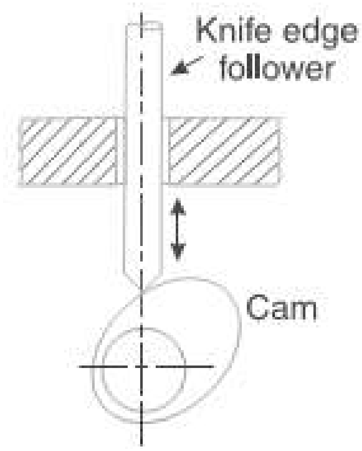
**Motion of the follower**
(i) Reciprocating or translating follower
(ii) Oscillating or rotating follower

**Path of motion of the follower**
(i) Radial follower
(ii) Offset follower

**1. According to the surface in contact:**

***a) Knife edge follower:***

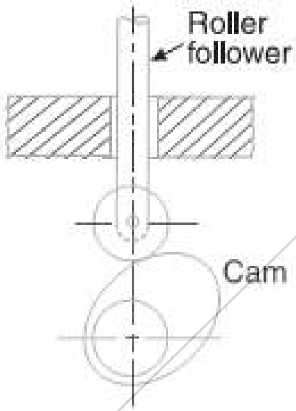
- When the contacting end of the follower has a sharp knife edge, it is called a knife edge follower.
- The sliding motion takes place between the contacting surfaces (i.e. the knife edge and the cam surface).
- It is seldom used in practice because the small area of contacting surface results in excessive wear.
- In knife edge followers, a considerable side thrust exists between the follower and the guide.




(*a*) Cam with knife edge follower.

***b) Roller follower:***

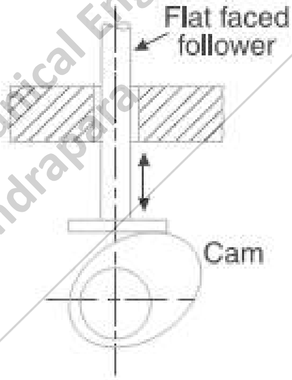
- When the contacting end of the follower is a roller, it is called a roller follower.
- Since the rolling motion takes place between the contacting surfaces (i.e. the roller and the cam), therefore the rate of wear is greatly reduced.
- In roller followers also the side thrust exists between the follower and the guide.
- The roller followers are extensively used where more space is available such as in stationary gas and oil engines and aircraft engines.



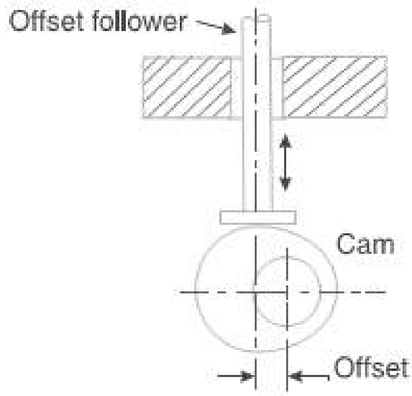

(*b*) Cam with roller follower.

***c) Flat faced or mushroom follower:***

- When the contacting end of the follower is a perfectly flat face, it is called a flat-faced follower.
- It may be noted that the side thrust between the follower and the guide is much reduced in case of flat faced followers. The only side thrust is due to friction between the contact surfaces of the follower and the cam.
- The relative motion between these surfaces is largely of sliding nature but wear may be reduced by off-setting the axis of the follower as shown in Fig., so that when the cam rotates, the follower also rotates about its own axis.
- The flat faced followers are generally used where space is limited such as in cams which operate the valves of automobile engines.




(*c*) Cam with flat faced follower.



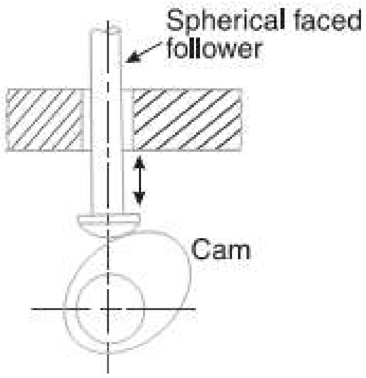

al (*f*) Cam with offset follower.



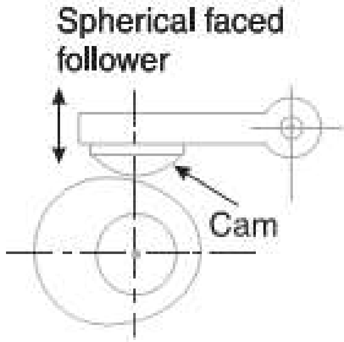
***Note:*** When the flat faced follower is circular, it is then called a mushroom follower.

***d) Spherical faced follower:***

- When the contacting end of the follower is of spherical shape, it is called a spherical faced follower.




(*d*) Cam with spherical faced follower.




(*e*) Cam with spherical faced follower.

- It may be noted that when a flat-faced follower is used in automobile engines, high surface stresses are produced. In order to minimize these stresses, the flat end of the follower is machined to a spherical shape.

**2. According to the motion of the follower:**

***a) Reciprocating or Translating follower:***

When the follower reciprocates in guides as the cam rotates uniformly, it is known reciprocating follower.

***b) Oscillating or Rotating follower:***

When the uniform rotary motion of the cam is connected into predetermined oscillatory motion of the follower, it is called oscillating or rotating follower.

**3. According to the motion of the follower:**

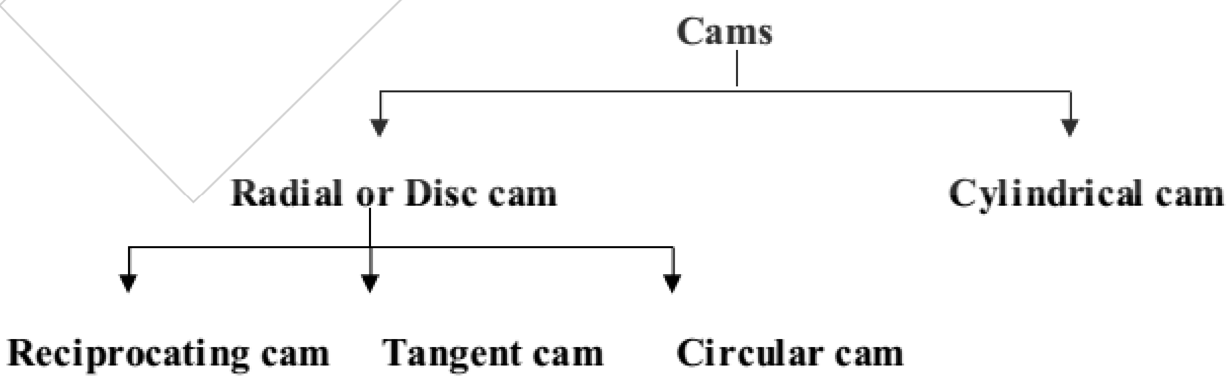
***a) Radial follower:***

When the motion of the follower is along an axis passing through the centre of the cam, it is known as radial follower.

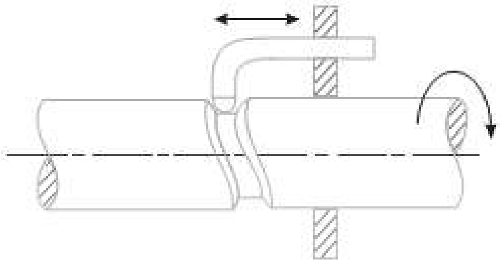
***b) Offset follower:***

When the motion of the follower is along an axis away from the axis from the centre, it is called off-set follower. Off-set can be done by springs, gravity or hydraulic means.

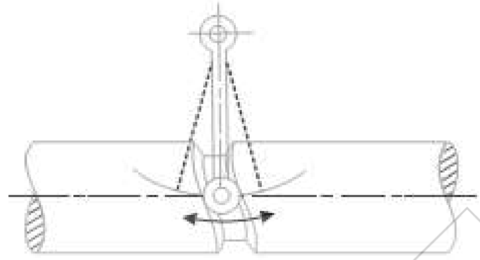
**Classification of Cams:**



***(a) Radial or Disc Cam:*** In radial cams, the follower reciprocates or oscillates in a direction perpendicular to the cam axis.



(*a*) Cylindrical cam with reciprocating follower.



(*b*) Cylindrical cam with oscillating follower.

***(b) Cylindrical Cam:*** In cylindrical cams, the follower reciprocates or oscillates in a direction parallel to the cam axis. The follower rides in a groove at its cylindrical surface.

# Chapter-02: Friction

**Friction:** When a body moves or tends to move on another body (surface/block), there exists some resistance or opposing force in a direction opposite to the direction of the movement of the body. This opposing force is called force of friction or friction.

**Types of Friction:**

1. ***Static friction:*** It is the friction, experienced by a body, when at rest.
2. ***Dynamic friction:*** It is the friction, experienced by a body, when in motion. The dynamic friction is also called ***kinetic friction*** and is less than the static friction.

a) ***Sliding friction:*** It is the friction, experienced by a body, when it slides over another body.

b) ***Rolling friction:*** It is the friction, experienced between the surfaces which have balls or rollers interposed between them.

c) ***Pivot friction:*** It is the friction, experienced by a body, due to the motion of rotation as in case of foot step bearings.

**The friction may further be classified as:**

1. ***Friction between un-lubricated surfaces:***
   - The friction experienced between two dry and un-lubricated surfaces in contact is known as ***dry or solid friction***.
   - It is due to the surface roughness.
   - The dry or solid friction includes the sliding friction and rolling friction as discussed above.
2. ***Friction between lubricated surfaces:*** When lubricant (i.e. oil or grease) is applied between two surfaces in contact, then the friction may be classified into the following two types depending upon the thickness of layer of a lubricant.

a) ***Boundary friction (or greasy friction or non-viscous friction):***

- It is the friction, experienced between the rubbing surfaces, when the surfaces have a very thin layer of lubricant.
- In this type of friction, a thin layer of lubricant forms a bond between the two rubbing surfaces.
- The lubricant is absorbed on the surfaces and forms a thin film. This thin film of the lubricant results in less friction between them. The boundary friction follows the laws of solid friction.

b) ***Fluid friction (or film friction or viscous friction):***

- It is the friction, experienced between the rubbing surfaces, when the surfaces have a thick layer of the lubricant.
- In this case, the actual surfaces do not come in contact and thus do not rub against each other.

- It is thus obvious that fluid friction is not due to the surfaces in contact but it is due to the viscosity and oiliness of the lubricant.

**Note:** The ***viscosity*** is a measure of the resistance offered to the sliding to one layer of the lubricant over an adjacent layer.

The ***oiliness*** property of a lubricant may be clearly understood by considering two lubricants of equal viscosities and at equal temperatures. When these lubricants are smeared on two different surfaces, it is found that the force of friction with one lubricant is different than that of the other. This difference is due to the property of the lubricant known as oiliness. ***The lubricant which gives lower force of friction is said to have greater oiliness***.

**Laws of friction:**

- When two bodies are in contact, the force of friction always acts in a direction opposite to that in which the body moves or tends to move.
- The force of friction is dependent upon the type of material of the bodies whose surfaces are in contact.
- The force of friction is independent of the area of contact between the two surfaces.
- The force of friction is independent of the relative velocity of sliding between the two surfaces.
- The force of frictions directly proportional to the normal reaction between the surfaces in contact for a particular material in contact for a particular material of which the bodies are made up.

$$F \propto N$$

$$\Rightarrow \quad F = \mu N$$

Where, $\mu$ = proportionality constant. i.e. also called as co-efficient of friction.

**Limiting Friction:**

When two bodies are in contact, the force of friction always acts in a direction opposite to that in which the body moves or tends to move.

Consider that a body A of weight W is lying on a rough horizontal body B as shown in Fig. 10.1 (a). In this position, the body A is in equilibrium under the action of its own weight W, and the normal reaction $R_N$ (equal to W) of B on A. Now if a small horizontal force P1 is applied to the body A acting through its centre of gravity as shown in Fig. 10.1 (b), it does not move because of the frictional force which prevents the motion. This shows that the applied force $P_1$ is exactly balanced by the force of friction $F_1$ acting in the opposite direction. If we now increase the applied force to $P_2$ as shown in Fig. 10.1 (c), it is still found to be in equilibrium. This means that the force of friction has also increased to a value $F_2 = P_2$.

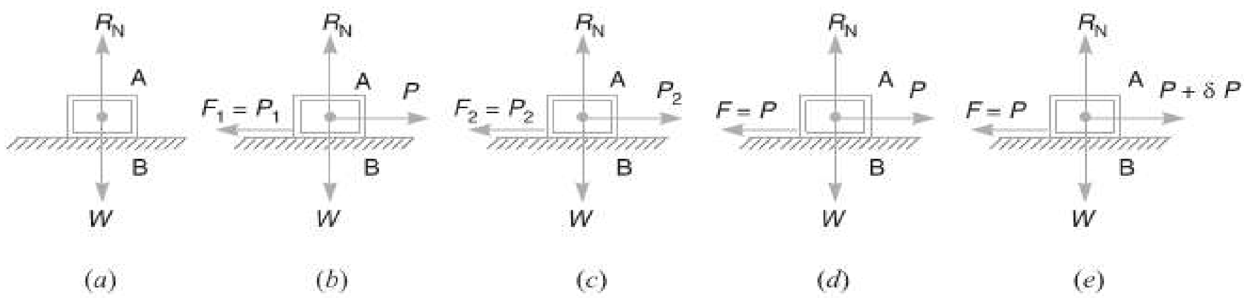


**Fig. 10.1.** Limiting friction.

Thus every time the effort is increased the force of friction also increases, so as to become exactly equal to the applied force. There is, however, a limit beyond which the force of friction cannot increase as shown in Fig. 10.1 (d). After this, any increase in the applied effort will not lead to any further increase in the force of friction, as shown in Fig. 10.1 (e), thus the body A begins to move in the direction of the applied force. *This maximum value of frictional force, which comes into play, when a body just begins to slide over the surface of the other body, is known as* ***limiting force of friction or simply limiting friction***.

It may be noted that when the applied force is less than the limiting friction, the body remains at rest, and the friction into play is called static friction which may have any value between zero and limiting friction.

**Limiting Angle of Friction:**

Consider that a body A of weight (W) is resting on a horizontal plane B, as shown in Fig. 10.2. If a horizontal force P is applied to the body, no relative motion will take place until the applied force P is equal to the force of friction F, acting opposite to the direction of motion. The magnitude of this force of friction is

$$F = \mu . W = \mu . R_N, \text{ where } R_N \text{ is the normal reaction.}$$

In the limiting case, when the motion just begins, the body will be in equilibrium under the action of the following three forces:

1. Weight of the body (W),
2. Applied horizontal force (P),

and 3. Reaction (R) between the body A and the plane B.

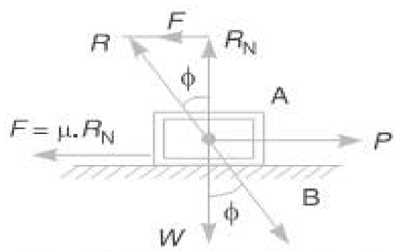


**Fig. 10.2.** Limiting angle of friction.

The reaction R must, therefore, be equal and opposite to the resultant of W and P and will be inclined at an angle φ to the normal reaction $R_N$. *This angle φ is known as the* ***limiting angle of friction****. It may be defined as the angle which the resultant reaction R makes with the normal reaction $R_N$.*

From Fig. 10.2, $\tan \varphi = F/R_N = \mu R_N / R_N = \mu$.

**Angle of Repose:**

Consider that a body A of weight (W) is resting on an inclined plane B, as shown in Fig. 10.3. If the angle of inclination α of the plane to the horizontal is such that the body begins to move down the plane, then the angle α is called the *angle of repose*. A little consideration will show that the body will begin to move down the plane when the angle of inclination of the plane is equal to the angle of friction (i.e. $\alpha = \varphi$).

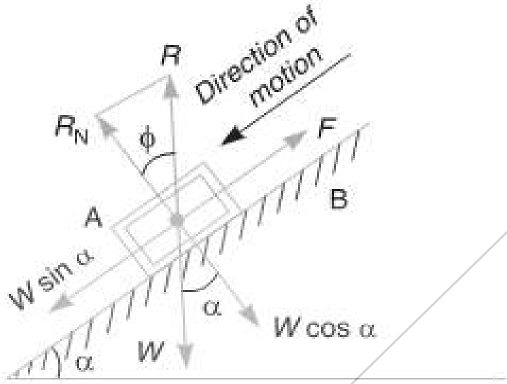



Fig. 10.3. Angle of repose.

The weight of the body (W) can be resolved into the following two components:

1. W sinα, parallel to the plane B.

This component tends to slide the body down the plane.

2. W cosα, perpendicular to the plane B. This component is balanced by the normal reaction ($R_N$) of the body A and the plane B.

The body will only begin to move down the plane, when

$$W \sin \alpha = F = \mu . R_N = \mu . W \cos \alpha \qquad \ldots (\because R_N = W \cos \alpha)$$

$$\therefore \qquad \tan \alpha = \mu = \tan \varphi \text{ or } \alpha = \varphi \qquad \ldots (\because \mu = \tan \varphi)$$

**Minimum Force Required to Slide a Body on a Rough Horizontal Plane:**

Consider that a body A of weight (W) is resting on a horizontal plane B as shown in Fig. 10.4. Let an effort P is applied at an angle θ to the horizontal such that the body A just moves.

$$P = \frac{W \sin \phi}{\cos (\theta - \phi)} \qquad \ldots (iii$$

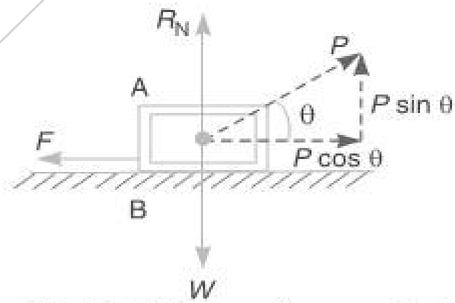



Fig. 10.4. Minimum force required to slide a body.

For P to be minimum, cos (θ – φ) should be maximum, i.e.

$$\cos (\theta - \varphi) = 1 \text{ or } \theta - \varphi = 0° \text{ or } \theta = \varphi$$

In other words, the effort P will be minimum, if its inclination with the horizontal is equal to the angle of friction.

$$\therefore \qquad P_{min} = W \sin \theta \qquad \ldots\text{[From equation (iii)]}$$

**Friction of a Body Lying on a Rough Inclined Plane:**

***1. Considering the motion of the body up the plane***

Let W = Weight of the body,

$\alpha$ = Angle of inclination of the plane to the horizontal,

$\varphi$ = Limiting angle of friction for the contact surfaces,

P = Effort applied in a given direction in order to cause the body to slide with uniform velocity parallel to the plane, considering friction,

$P_0$ = Effort required to move the body up the plane neglecting friction,

$\theta$ = Angle which the line of action of P makes with the weight of the body W,

$\mu$ = Coefficient of friction between the surfaces of the plane and the body,

$R_N$ = Normal reaction, and R = Resultant reaction.

When the friction is neglected, the body is in equilibrium under the action of the three forces, i.e. $P_0$, W and $R_N$, as shown in Fig. 10.7 (a). The triangle of forces is shown in Fig. 10.7 (b). Now applying sine rule for these three concurrent forces,

$$\frac{P_0}{\sin\alpha} = \frac{W}{\sin(\theta - \alpha)} \quad \text{or} \quad P_0 = \frac{W \sin\alpha}{\sin(\theta - \alpha)} \qquad ...(i)$$

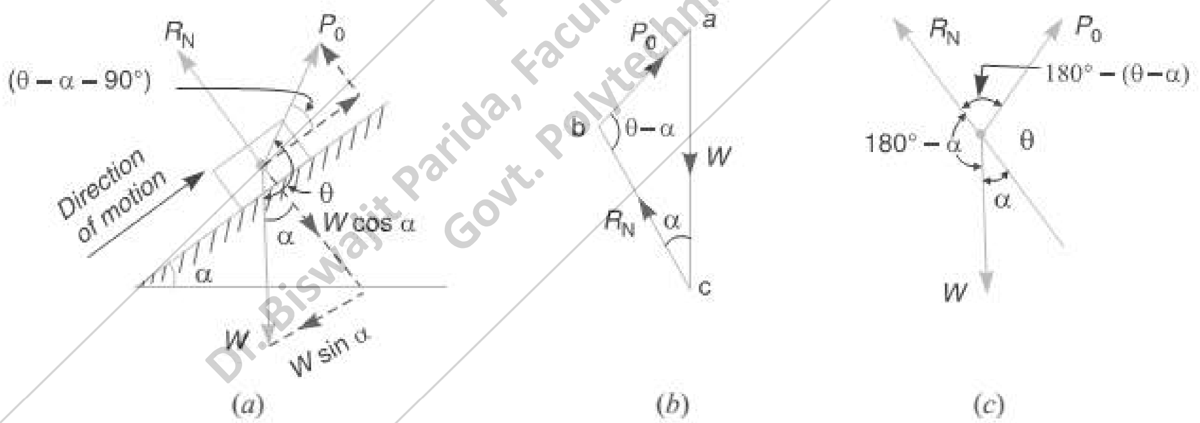



**Fig. 10.7.** Motion of the body up the plane, neglecting friction.

When friction is taken into account, a frictional force F= $\mu.R_N$ acts in the direction opposite to the motion of the body, as shown in Fig. 10.8 (a). The resultant reaction R between the plane and the body is inclined at an angle $\varphi$ with the normal reaction $R_N$. The triangle of forces is shown in Fig. 10.8 (b). Now applying sine rule,

$$\frac{P}{\sin(\alpha + \phi)} = \frac{W}{\sin[\theta - (\alpha + \phi)]}$$

$$\therefore \quad P = \frac{W \sin(\alpha + \phi)}{\sin[\theta - (\alpha + \phi)]} \qquad ...(ii)$$

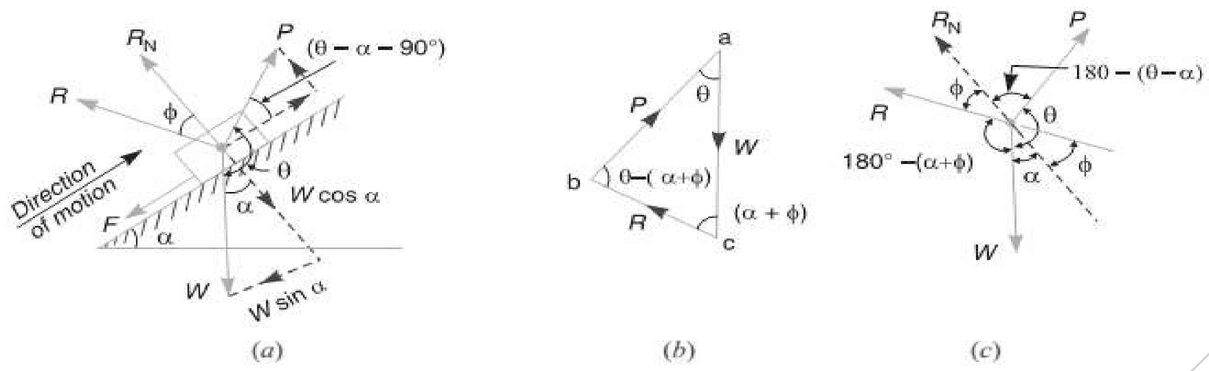



**Fig. 10.8.** Motion of the body up the plane, considering friction.

**Notes : 1.** When the effort applied is horizontal, then $\theta = 90°$. In that case, the equations **(*i*)** and **(*ii*)** may be written as

$$P_0 = \frac{W \sin \alpha}{\sin (90° - \alpha)} = \frac{W \sin \alpha}{\cos \alpha} = W \tan \alpha$$

and

$$P = \frac{W \sin (\alpha + \phi)}{\sin [90° - (\alpha + \phi)]} = \frac{W \sin (\alpha + \phi)}{\cos (\alpha + \phi)} = W \tan (\alpha + \phi)$$

**2.** When the effort applied is parallel to the plane, then $\theta = 90° + \alpha$. In that case, the equations **(*i*)** and **(*ii*)** may be written as

$$P_0 = \frac{W \sin \alpha}{\sin (90° + \alpha - \alpha)} = W \sin \alpha$$

and

$$P = \frac{W \sin (\alpha + \phi)}{\sin [(90° + \alpha) - (\alpha + \phi)]} = \frac{W \sin (\alpha + \phi)}{\cos \phi}$$

$$= \frac{W (\sin \alpha \cos \phi + \cos \alpha \sin \phi)}{\cos \phi} = W (\sin \alpha + \cos \alpha . \tan \phi)$$

$$= W (\sin \alpha + \mu \cos \alpha) \qquad ...(\because \mu = \tan \phi)$$

**2.** ***Considering the motion of the body down the plane***

Neglecting friction, the effort required for the motion down the plane will be same as for the motion up the plane, *i.e.*

$$P_0 = \frac{W \sin \alpha}{\sin (\theta - \alpha)} \qquad ...(iii)$$

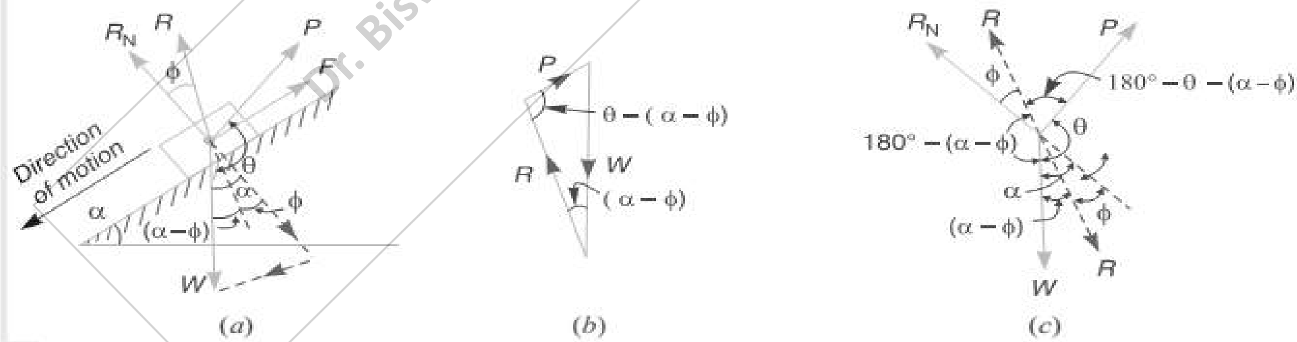



**Fig. 10.9.** Motion of the body down the plane, considering friction.

When the friction is taken into account, the force of friction $F = \mu . R_N$ will act up the plane and the resultant reaction $R$ will make an angle $\phi$ with $R_N$ towards its right as shown in Fig. 10.9 (*a*). The triangle of forces is shown in Fig. 10.9 (*b*). Now from sine rule,

$$\frac{P}{\sin (\alpha - \phi)} = \frac{W}{\sin [\theta - (\alpha - \phi)]}$$

or

$$P = \frac{W \sin (\alpha - \phi)}{\sin [\theta - (\alpha - \phi)]} \qquad ...(iv)$$

**Notes : 1.** The value of $P$ may also be obtained either by applying Lami's theorem to Fig. 10.9 ($c$), or by resolving the forces along the plane and perpendicular to the plane and then using $\Sigma H = 0$ and $\Sigma V = 0$ (See Art. 10.18 and 10.19).

**2.** When $P$ is applied horizontally, then $\theta = 90°$. In that case, equation ($iv$) may be written as

$$P = \frac{W \sin(\alpha - \phi)}{\sin[90° - (\alpha - \phi)]} = \frac{W \sin(\alpha - \phi)}{\cos(\alpha - \phi)} = W \tan(\alpha - \phi))$$

**3.** When $P$ is applied parallel to the plane, then $\theta = 90° + \alpha$. In that case, equation (***iv***) may be written as

$$P = \frac{W \sin(\alpha - \phi)}{\sin[90° + \alpha) - (\alpha - \phi)]} = \frac{W \sin(\alpha - \phi)}{\cos \phi}$$

$$= \frac{W(\sin\alpha \cos\phi - \cos\alpha \sin\phi)}{\cos\phi} = W(\sin\alpha - \tan\phi \cos\alpha)$$

$$= W(\sin\alpha - \mu \cos\alpha) \qquad ...(\because \tan\phi = \mu)$$

**Efficiency of Inclined Plane:**

The ratio of the effort required neglecting friction (i.e. $P_0$) to the effort required considering friction (i.e. P) is known as efficiency of the inclined plane. Mathematically, efficiency of the inclined plane,

$$\eta = P_0 / P$$

Let us consider the following two cases:

**1.** ***For the motion of the body up the plane***

Efficiency, $$\eta = \frac{P_0}{P} = \frac{W \sin\alpha}{\sin(\theta - \alpha)} \times \frac{\sin[\theta - (\alpha + \phi)]}{W \sin(\alpha + \phi)}$$

$$= \frac{\sin\alpha}{\sin\theta \cos\alpha - \cos\theta \sin\alpha} \times \frac{\sin\theta \cos(\alpha + \phi) - \cos\theta \sin(\alpha + \phi)}{\sin(\alpha + \phi)}$$

Multiplying the numerator and denominator by $\sin(\alpha + \phi) \sin\theta$, we get

$$\eta = \frac{\cot(\alpha + \phi) - \cot\theta}{\cot\alpha - \cot\theta}$$

**Notes : 1.** When effort is applied horizontally, then $\theta = 90°$.

$$\therefore \qquad \eta = \frac{\tan\alpha}{\tan(\alpha + \phi)}$$

**2.** When effort is applied parallel to the plane, then $\theta = 90° + \alpha$.

$$\therefore \qquad \eta = \frac{\cot(\alpha + \phi) - \cot(90° + \alpha)}{\cot\alpha - \cot(90° + \alpha)} = \frac{\cot(\alpha + \phi) + \tan\alpha}{\cot\alpha + \tan\alpha} = \frac{\sin\alpha \cos\phi}{\sin(\alpha + \phi)}$$

**2.** ***For the motion of the body down the plane***

Since the value of $P$ will be less than $P_0$, for the motion of the body down the plane, therefore in this case,

$$\eta = \frac{P}{P_0} = \frac{W \sin(\alpha - \phi)}{\sin[\theta - (\alpha - \phi)]} \times \frac{\sin(\theta - \alpha)}{W \sin\alpha}$$

$$= \frac{\sin(\alpha - \phi)}{\sin\theta \cos(\alpha - \phi) - \cos\theta \sin(\alpha - \phi)} \times \frac{\sin\theta \cos\alpha - \cos\theta \sin\alpha}{\sin\alpha}$$

Multiplying the numerator and denominator by $\sin(\alpha - \phi)\sin\theta$, we get

$$\eta = \frac{\cot\alpha - \cot\theta}{\cot(\alpha - \phi) - \cot\theta}$$

**Notes : 1.** When effort is applied horizontally, then $\theta = 90^\circ$.

$$\therefore \quad \eta = \frac{\cot\alpha}{\cot(\alpha - \phi)} = \frac{\tan(\alpha - \phi)}{\tan\alpha}$$

**2.** When effort is applied parallel to the plane, then $\theta = 90^\circ + \alpha$.

$$\therefore \quad \eta = \frac{\cot\alpha - \cot(90^\circ + \alpha)}{\cot(\alpha - \phi) - \cot(90^\circ + \alpha)} = \frac{\cot\alpha + \tan\alpha}{\cot(\alpha - \phi) + \tan\alpha} = \frac{\sin(\alpha - \phi)}{\sin\alpha\cos\phi}$$

## Screw Friction:

- The screws, bolts, studs, nuts etc. are widely used in various machines and structures for temporary fastenings.
- These fastenings have screw threads, which are made by cutting a continuous helical groove on a cylindrical surface.
- If the threads are cut on the outer surface of a solid rod, these are known as external threads. But if the threads are cut on the internal surface of a hollow rod, these are known as internal threads.
- The screw threads are mainly of two types i.e. V-threads and square threads. The V-threads are stronger and offer more frictional resistance to motion than square threads. Moreover, the V-threads have an advantage of preventing the nut from slackening.
- In general, the V-threads are used for the purpose of tightening pieces together e.g. bolts and nuts etc. But the square threads are used in screw jacks, vice screws etc. The following terms are important for the study of screw:

1. ***Helix:*** It is the curve traced by a particle, while describing a circular path at a uniform speed and advancing in the axial direction at a uniform rate. In other words, it is the curve traced by a particle while moving along a screw thread.
2. ***Pitch:*** It is the distance from a point of a screw to a corresponding point on the next thread, measured parallel to the axis of the screw.
3. ***Lead:*** It is the distance from a point of a screw to a corresponding point on the next thread, measured parallel to the axis of the screw.
4. ***Depth of thread:*** It is the distance between the top and bottom surfaces of a thread (also known as crest and root of a thread).
5. ***Single-threaded screw:*** If the lead of a screw is equal to its pitch, it is known as single threaded screw.
6. ***Multi-threaded screw:*** If more than one thread is cut in one lead distance of a screw, it is known as multi-threaded screw e.g. in a double threaded screw, two threads are cut in one lead length.
   In such cases, all the threads run independently along the length of the rod.
   Mathematically, Lead = Pitch × Number of threads.

7. ***Helix angle:*** It is the slope or inclination of the thread with the horizontal. Mathematically.

$$\tan \alpha = \frac{\text{Lead of screw}}{\text{Circumference of screw}}$$

$$= p/\pi d \quad \text{...(In single-threaded screw)}$$

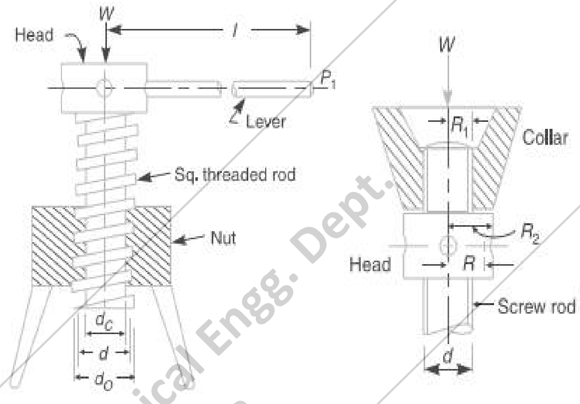
$$= n.p/\pi d \quad \text{...(In multi-threaded screw)}$$

Where, α = Helix angle, p = Pitch of the screw, d = Mean diameter of the screw, and n = Number of threads in one lead.

**Screw Friction:**

The screw jack is a device, for lifting heavy loads, by applying a comparatively smaller effort at its handle.

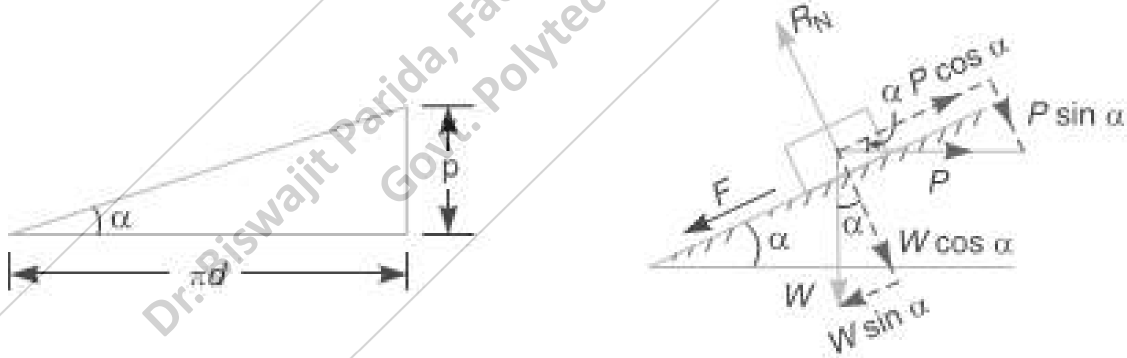
The load, to be raised or lowered, is placed on the head of the square threaded rod which is rotated by the application of an effort at the end of the lever for lifting or lowering the load.




(*a*) Screw jack. (*b*) Thrust collar.

**Torque Required to Lift the Load by a Screw Jack:**

If one complete turn of a screw thread by imagined to be unwound, from the body of the screw and developed, it will form an inclined plane as shown in Fig. 10.12 (a).




(*a*) Development of a screw. (*b*) Forces acting on the screw.

Let p = Pitch of the screw,

d = Mean diameter of the screw,

α = Helix angle,

P = Effort applied at the circumference of the screw to lift the load,

W = Load to be lifted, and

μ = Coefficient of friction, between the screw and nut = tan φ,

where, φ is the friction angle.

From the geometry of the Fig. 10.12 (a), we find that

$$\tan \alpha = p/\pi d$$

Since the principle on which a screw jack works is similar to that of an inclined plane, therefore the force applied on the lever of a screw jack may be considered to be horizontal as shown in Fig. 10.12 (b).

Since the load is being lifted, therefore the force of friction ($F = \mu.R_N$) will act downwards.

Resolving the forces along the plane,

$$P \cos \alpha = W \sin \alpha + F = W \sin \alpha + \mu.R_N \qquad ...(i)$$

and resolving the forces perpendicular to the plane,

$$R_N = P \sin \alpha + W \cos \alpha \qquad ...(ii)$$

Substituting this value of $R_N$ in equation (i),

$$P \cos \alpha = W \sin \alpha + \mu (P \sin \alpha + W \cos \alpha)$$

$$= W \sin \alpha + \mu P \sin \alpha + \mu W \cos \alpha$$

or $$P \cos \alpha - \mu P \sin \alpha = W \sin \alpha + \mu W \cos \alpha$$

or $$P (\cos \alpha - \mu \sin \alpha) = W (\sin \alpha + \mu \cos \alpha)$$

$$\therefore \quad P = W \times \frac{\sin \alpha + \mu \cos \alpha}{\cos \alpha - \mu \sin \alpha}$$

Substituting the value of $\mu = \tan \phi$ in the above equation, we get

$$P = W \times \frac{\sin \alpha + \tan \phi \cos \alpha}{\cos \alpha - \tan \phi \sin \alpha}$$

Multiplying the numerator and denominator by $\cos \phi$,

$$P = W \times \frac{\sin \alpha \cos \phi + \sin \phi \cos \alpha}{\cos \alpha \cos \phi - \sin \alpha \sin \phi} = W \times \frac{\sin (\alpha + \phi)}{\cos (\alpha + \phi)}$$

$$= W \tan (\alpha + \phi)$$

$\therefore$ Torque required to overcome friction between the screw and nut,

$$T_1 = P \times \frac{d}{2} = W \tan (\alpha + \phi) \frac{d}{2}$$

When the axial load is taken up by a thrust collar or a flat surface, as shown in Fig. 10.11 (*b*), so that the load does not rotate with the screw, then the torque required to overcome friction at the collar,

$$T_2 = \mu_1 W \left( \frac{R_1 + R_2}{2} \right) = \mu_1 W.R$$

where $R_1$ and $R_2$ = Outside and inside radii of the collar, $R$ = Mean radius of the collar, and $\mu_1$ = Coefficient of friction for the collar.

∴ Total torque required to overcome friction (*i.e.* to rotate the screw),

$$T = T_1 + T_2 = P \times \frac{d}{2} + \mu_1 W.R$$

If an effort $P_1$ is applied at the end of a lever of arm length $l$, then the total torque required to overcome friction must be equal to the torque applied at the end of the lever, *i.e.*

$$T = P \times \frac{d}{2} = P_1 l$$

**Notes : 1.** When the *nominal diameter ($d_0$) and the **core diameter ($d_c$) of the screw thread is given, then the mean diameter of the screw,

$$d = \frac{d_0 + d_c}{2} = d_0 - \frac{p}{2} = d_c + \frac{p}{2}$$

**2.** Since the mechanical advantage is the ratio of load lifted ($W$) to the effort applied ($P_1$) at the end of the lever, therefore mechanical advantage,

$$M.A. = \frac{W}{P_1} = \frac{W \times 2l}{p.d} \qquad \ldots \left( \because P_1 = \frac{P.d}{2l} \right)$$

$$= \frac{W \times 2l}{W \tan(\alpha + \phi)\, d} = \frac{2l}{d.\tan(\alpha + \phi)}$$

**Example 10.3.** *An electric motor driven power screw moves a nut in a horizontal plane against a force of 75 kN at a speed of 300 mm/min. The screw has a single square thread of 6 mm pitch on a major diameter of 40 mm. The coefficient of friction at the screw threads is 0.1. Estimate power of the motor.*

* The **nominal diameter** of a screw thread is also known as **outside diameter** or **major diameter.**

** The **core diameter** of a screw thread is also known as **inner diameter** or **root diameter** or **minor diameter.**

**Solution.** Given : $W$ = 75 kN = 75 × 10³ N ; $v$ = 300 mm/min ; $p$ = 6 mm ; $d_0$ = 40 mm ; $\mu = \tan \phi = 0.1$

We know that mean diameter of the screw,

$$d = d_0 - p/2 = 40 - 6/2 = 37 \text{ mm} = 0.037 \text{ m}$$

and

$$\tan \alpha = \frac{p}{\pi d} = \frac{6}{\pi \times 37} = 0.0516$$

∴ Force required at the circumference of the screw,

$$P = W \tan(\alpha + \phi) = W \left[ \frac{\tan \alpha + \tan \phi}{1 - \tan \alpha . \tan \phi} \right]$$

$$= 75 \times 10^3 \left[ \frac{0.0516 + 0.1}{1 - 0.0516 \times 0.1} \right] = 11.43 \times 10^3 \text{ N}$$

and torque required to overcome friction,

$$T = P \times d/2 = 11.43 \times 10^3 \times 0.037/2 = 211.45 \text{ N-m}$$

We know that speed of the screw,

$$N = \frac{\text{Speed of the nut}}{\text{Pitch of the screw}} = \frac{300}{6} = 50 \text{ r.p.m.}$$

and angular speed, $\omega = 2\pi \times 50/60 = 5.24$ rad/s

∴ Power of the motor $= T.\omega = 211.45 \times 5.24 = 1108$ W = 1.108 **kW Ans.**

**Example 10.5.** *A 150 mm diameter valve, against which a steam pressure of 2 MN/m² is acting, is closed by means of a square threaded screw 50 mm in external diameter with 6 mm pitch. If the coefficient of friction is 0.12 ; find the torque required to turn the handle.*

**Solution.** Given : $D$ = 150 mm = 0.15 mm = 0.15 m ; $Ps$ = 2 MN/m² = 2 × $10^6$ N/m² ; $d_0$ = 50 mm ; $p$ = 6 mm ; $\mu = \tan\phi = 0.12$

We know that load on the valve,

$$W = \text{Pressure} \times \text{Area} = p_S \times \frac{\pi}{4} D^2 = 2 \times 10^6 \times \frac{\pi}{4}(0.15)^2 \text{ N}$$
$$= 35\ 400 \text{ N}$$

Mean diameter of the screw,

$$d = d_0 - p/2 = 50 - 6/2 = 47 \text{ mm} = 0.047 \text{ m}$$

$$\therefore \quad \tan\alpha = \frac{p}{\pi d} = \frac{6}{\pi \times 47} = 0.0406$$

We know that force required to turn the handle,

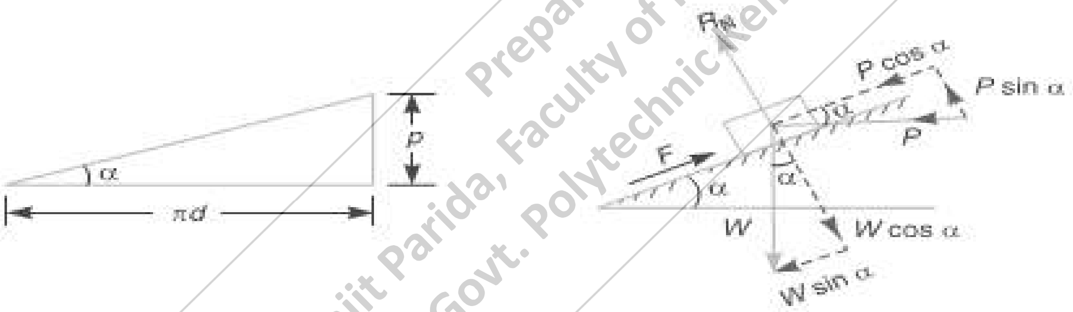
$$P = W \tan(\alpha + \phi) = W\left[\frac{\tan\alpha + \tan\phi}{1 - \tan\alpha . \tan\phi}\right]$$

$$= 35400\left[\frac{0.0406 + 12}{1 - 0.0406 \times 0.12}\right] = 5713 \text{ N}$$

∴ Torque required to turn the handle,

$$T = P \times d/2 = 5713 \times 0.047/2 = 134.2 \text{ N-m} \quad \textbf{Ans.}$$

## Torque Required to Lower the Load by a Screw Jack:




(*a*) (*b*)

**Fig. 10.13**

$$\therefore \quad P = W \times \frac{(\mu \cos\alpha - \sin\alpha)}{(\cos\alpha + \mu \sin\alpha)}$$

Substituting the value of $\mu = \tan\phi$ in the above equation, we get

$$P = W \times \frac{(\tan\phi \cos\alpha - \sin\alpha)}{(\cos\alpha + \tan\phi \sin\alpha)}$$

Multiplying the numerator and denominator by cos φ,

$$P = W \times \frac{(\sin\phi \cos\alpha - \sin\alpha \cos\phi)}{(\cos\alpha \cos\phi + \sin\phi \sin\alpha)} = W \times \frac{\sin(\phi - \alpha)}{\cos(\phi - \alpha)}$$
$$= W \tan(\phi - \alpha)$$

∴ Torque required to overcome friction between the screw and nut,

$$T = P \times \frac{d}{2} = W \tan(\phi - \alpha)\frac{d}{2}$$

**Note :** When $\alpha > \phi$, then $P = \tan(\alpha - \phi)$.

**Example 10.9.** *The mean diameter of a square threaded screw jack is 50 mm. The pitch of the thread is 10 mm. The coefficient of friction is 0.15. What force must be applied at the end of a 0.7 m long lever, which is perpendicular to the longitudinal axis of the screw to raise a load of 20 kN and to lower it?*

**Solution.** Given : $d = 50$ mm $= 0.05$ m ; $p = 10$ mm ; $\mu = \tan\phi = 0.15$ ; $l = 0.7$ m ; $W = 20$ kN $= 20 \times 10^3$ N

We know that $\tan\alpha = \dfrac{p}{\pi d} = \dfrac{10}{\pi \times 50} = 0.0637$

Let $P_1$ = Force required at the end of the lever.

*Force required to raise the load*

We know that force required at the circumference of the screw,

$$P = W\tan(\alpha + \phi) = W\left[\frac{\tan\alpha + \tan\phi}{1 - \tan\alpha.\tan\phi}\right]$$

$$= 20 \times 10^3\left[\frac{0.0637 + 0.15}{1 - 0.0637 \times 0.15}\right] = 4314 \text{ N}$$

Now the force required at the end of the lever may be found out by the relation,

$$P_1 \times l = P \times d/2$$

$$\therefore \quad P_1 = \frac{P \times d}{2\,l} = \frac{4314 \times 0.05}{2 \times 0.7} = 154 \text{ N} \quad \textbf{Ans.}$$

*Force required to lower the load*

We know that the force required at the circumference of the screw,

$$P = W\tan(\phi - \alpha) = W\left[\frac{\tan\phi - \tan\alpha}{1 + \tan\phi.\tan\alpha}\right]$$

$$= 20 \times 10^3\left[\frac{0.15 - 0.0637}{1 + 0.15 \times 0.0637}\right] = 1710 \text{ N}$$

Now the force required at the end of the lever may be found out by the relation,

$$P_1 \times l = P \times \frac{d}{2} \quad \text{or} \quad P_1 = \frac{P \times d}{2l} = \frac{1710 \times 0.05}{2 \times 0.7} = 61 \text{ N} \quad \textbf{Ans.}$$



## Efficiency of a Screw Jack:

The efficiency of a screw jack may be defined as **the ratio between the ideal effort** (*i.e.* the effort required to move the load, neglecting friction) to **the actual effort** (*i.e.* the effort required to move the load taking friction into account).

We know that the effort required to lift the load ($W$) when friction is taken into account,

$$P = W\tan(\alpha + \phi) \qquad ...(i)$$

where $\alpha$ = Helix angle,

$\phi$ = Angle of friction, and

$\mu$ = Coefficient of friction, between the screw and nut $= \tan\phi$.

If there would have been no friction between the screw and the nut, then $\phi$ will be equal to zero. The value of effort $P_0$ necessary to raise the load, will then be given by the equation,

$$P_0 = W\tan\alpha \qquad [\textit{i.e.} \text{ Putting } \phi = 0 \text{ in equation } (i)]$$

$$\therefore \text{ Efficiency, } \eta = \frac{\text{Ideal effort}}{\text{Actual effort}} = \frac{P_0}{P} = \frac{W\tan\alpha}{W\tan(\alpha+\phi)} = \frac{\tan\alpha}{\tan(\alpha+\phi)}$$

which shows that the efficiency of a screw jack, is independent of the load raised.

In the above expression for efficiency, only the screw friction is considered. However, if the screw friction and the collar friction is taken into account, then

$$\therefore \quad \eta = \frac{\text{Torque required to move the load, neglecting friction}}{\text{Torque required to move the load, including screw and collar friction}}$$

$$= \frac{T_0}{T} = \frac{P_0 \times d/2}{P \times d/2 + \mu_1 . W . R}$$

**Note:** The efficiency of the screw jack may also be defined as **the ratio of mechanical advantage to the velocity ratio.**

We know that mechanical advantage,

$$M.A. = \frac{W}{P_1} = \frac{W \times 2l}{P \times d} = \frac{W \times 2l}{W \tan(\alpha + \phi) d} = \frac{2l}{\tan(\alpha + \phi) d} \qquad \text{...(Refer Art 10.17)}$$

and velocity ratio,

$$V.R. = \frac{\text{Distance moved by the effort } (P_1) \text{, in one revolution}}{\text{Distance moved by the load } (W) \text{, in one revolution}}$$

$$= \frac{2\pi l}{p} = \frac{2\pi l}{\tan\alpha \times \pi d} = \frac{2l}{\tan\alpha \times d} \qquad \text{...}(\because \tan\alpha = p/\pi d)$$

$$\therefore \text{ Efficiency, } \eta = \frac{M.A.}{V.R.} = \frac{2l}{\tan(\alpha+\phi) d} \times \frac{\tan\alpha \times d}{2l} = \frac{\tan \times \alpha}{\tan(\alpha + \phi)}$$

## Maximum Efficiency of a Screw Jack:

We have seen in Art. 10.20 that the efficiency of a screw jack,

$$\eta = \frac{\tan\alpha}{\tan(\alpha+\theta)} = \frac{\dfrac{\sin\alpha}{\cos\alpha}}{\dfrac{\sin(\alpha+\phi)}{\cos(\alpha+\phi)}} = \frac{\sin\alpha \times \cos(\alpha+\phi)}{\cos\alpha \times \sin(\alpha+\phi)} \qquad \text{...}(i)$$

$$= \frac{2\sin\alpha \times \cos(\alpha+\phi)}{2\cos\alpha \times \sin(\alpha+\phi)}$$

...(Multiplying the numerator and denominator by 2)

$$= \frac{\sin(2\alpha+\phi) - \sin\phi}{\sin(2\alpha+\phi) + \sin\phi} \qquad \text{...}(ii)$$

$$\text{...}\left[\begin{array}{l} \because\ 2\sin A \cos B = \sin(A+B) + \sin(A-B) \\ \quad 2\cos A \sin B = \sin(A+B) - \sin(A-B) \end{array}\right]$$

The efficiency given by equation (*ii*) is maximum when $\sin(2\alpha + \phi)$ is maximum, *i.e.* when

$$\sin(2\alpha+\phi) = 1 \quad \text{or} \quad \text{when } 2\alpha + \phi = 90^\circ$$

$$\therefore \qquad 2\alpha = 90^\circ - \phi \quad \text{or} \quad \alpha = 45^\circ - \phi/2$$

Substituting the value of $2\alpha$ in equation (*ii*), we have maximum efficiency,

$$\eta_{max} = \frac{\sin(90^\circ - \phi + \phi) - \sin\phi}{\sin(90^\circ - \phi + \phi) + \sin\phi} = \frac{\sin 90^\circ - \sin\phi}{\sin 90^\circ + \sin\phi} = \frac{1 - \sin\phi}{1 + \sin\phi}$$

**Example 10.10.** *The pitch of 50 mm mean diameter threaded screw of a screw jack is 12.5 mm. The coefficient of friction between the screw and the nut is 0.13. Determine the torque required on the screw to raise a load of 25 kN, assuming the load to rotate with the screw. Determine the ratio of the torque required to raise the load to the torque required to lower the load and also the efficiency of the machine.*

**Solution.** Given : $d = 50$ mm ; $p = 12.5$ mm ; $\mu = \tan\phi = 0.13$ ; $W = 25$ kN $= 25 \times 10^3$ N

We know that, $\tan\alpha = \dfrac{p}{\pi d} = \dfrac{12.5}{\pi \times 50} = 0.08$

and force required on the screw to raise the load,

$$P = W\tan(\alpha + \phi) = W\left[\frac{\tan\phi - \tan\alpha}{1 + \tan\phi . \tan\alpha}\right]$$

$$= 25 \times 10^3\left[\frac{0.08 + 0.13}{1 - 0.08 \times 0.13}\right] = 5305 \text{ N}$$

***Torque required on the screw***

We know that the torque required on the screw to raise the load,

$$T_1 = P \times d/2 = 5305 \times 50/2 = 132\,625 \text{ N-mm} \textbf{ Ans.}$$

***Ratio of the torques required to raise and lower the load***

We know that the force required on the screw to lower the load,

$$P = W\tan(\phi - \alpha) = W\left[\frac{\tan\phi - \tan\alpha}{1 + \tan\phi . \tan\alpha}\right]$$

$$= 25 \times 10^3\left[\frac{0.13 + 0.08}{1 + 0.13 \times 0.08}\right] = 1237 \text{ N}$$

and torque required to lower the load

$$T_2 = P \times d/2 = 1237 \times 50/2 = 30\,905 \text{ N-mm}$$

$\therefore$ Ratio of the torques required,

$$= T_1 / T_2 = 132625/30925 = 4.3 \textbf{ Ans.}$$

***Efficiency of the machine***

We know that the efficiency,

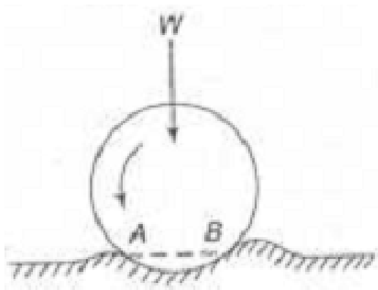
$$\eta = \frac{\tan\alpha}{\tan(\alpha + \phi)} = \frac{\tan\alpha(1 - \tan\alpha . \tan\phi)}{\tan\alpha + \tan\phi} = \frac{0.08(1 - 0.08 \times 0.13)}{0.08 + 0.13}$$

$$= 0.377 = 37.7\% \textbf{ Ans.}$$



## Rolling friction:

When a cylinder rolls over a flat surface, it makes a line contact parallel to the axis of the cylinder, and when a sphere roll, over a flat surface it makes a point contact there is no sliding at the point or line contact there is no sliding at the po6int or line contact.




## Bearings:

The word bearing is derived from the verb "to bear". A bearing is a machine element that allows one part to bear another. Bearing is a mechanical element that permits relative motion between the two parts, such as the craft and the housing with minimum friction.

The main *functions* of bearings are

- The bearing ensures free rotation of the shaft or the angle with minimum friction.

- To support rotating parts (the shaft or the axle) of a machine and holds it in the correct position.
- To bear radial and thrust load that act on the shaft or the axle and transmits them to the frame and foundation.

The main ***applications*** of bearings are

***Rolling contact bearings***

- Machine tool spindles
- Automobile front or rear axles
- Gear boxes
- Small size electric motors
- Rope sleeves, crane hooks and hoisting drums.

***Sliding contact bearings***

- Crank shaft bearing in diesel and petrol engines
- Steam and gas turbines.
- Concrete mixtures
- Rope conveyors

**Classification of Bearings:**

**A. According to Friction:**

**1. Friction bearing:**

- As the name implies, in this bearings the bearing surface is in contact with moving surface or the shaft which produces more friction.
- These bearing are made up of cast iron, bronze, brass, babbit and white metal having hollow round shape.
- Lubricant is used for slow moving and heavy weighted running on shaft.
- These bearings to support crack shaft, rocker arm of IC engine.

***a) Solid bearings:***

- These are made up of cast iron or bronze in the form of bush and press-fitted in fabricated or cast iron housings.
- This is used for small and light shafts moving at low speed.
- A hole is provided on its face of lubrication.

***b) Split bearings:***

- Split bearings have an arrangement of split.
- Split bearings are made in halves and assembled in special plumber blocks.
- It has collar on its external surfaces and also made in two parts.

*c) Self-aligning bush bearings:*

- It consist mainly two parts. The first one is cast iron block and other is bush.
- These bearing bush are made up of brass or any other soft material in round shape.
- To protect it from moving, a screw is fixed at one end and this screw is fixed half to the bush and half in block.

*d) Adjustable slide bearings:*

- It can adjust the tightness between bearings and the shaft.
- This type of bearing has provision for wear adjustment.
- The bearing is fitted in the tapered hole of the housing for adjustment of wear.
- The bearing is drawn inside by means of a slotted ring nut.

**Advantages of Friction bearing:**

- Friction bearings are cheap to produce and have noiseless operation.
- They can be easily machined, occupy small radial space and have vibration damping properties.
- They can cope with tapped to the foreign matter.

**Disadvantages of Friction bearing:**

- It damages machines.
- It restricts early movement of machine
- It produces a lot of heat energy.



**2. Anti-friction bearing:**

- The main purpose of these bearings is to minimize the friction in bearing.
- Due to this reason, the speed of an object increases and friction and temperature decreases.
- Such bearing have long life.

***a) Ball bearings:***

A ball bearing is a rolling element bearing that use ball to maintain the separation between the bearing races.

***i.)* Single row Ball bearings:**

These bearing have only one groove cut in outer and inner rings with the ball in identical line.

***ii.)* Double row Ball bearings:**

These bearing have two grooves cut in inner and outer ring lie in two rows of the bearing.

***iii.)*Self-aligning Ball bearings:**

These bearings can withstand with journal loads. These loads are generally inclined due to shaft misalignment. These types of bearings have a spherical bore on the outer race.

***iv.)* Angular contact Ball bearings:**

These bearings are designed to take an axial thrust as well as radial loads.

***v.)* Thrust Ball bearings:**

These bearings are useful for taking vertical thrust load but cannot take any radial load. Some special thrust bearings are available which can also take horizontal end thrust.

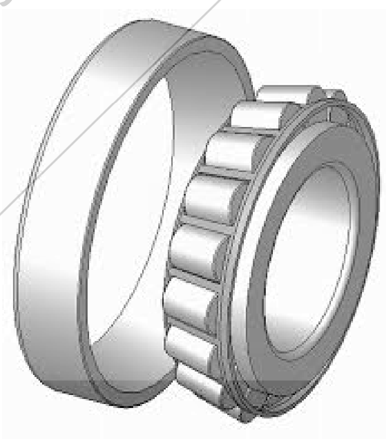
***b) Roller bearings:***

- Roller bearings are available with the grooved race in the outer and inner members.
- Roller bearings are capable of carrying the journal (radial) loads.
- It can work with greater load than ball bearings.

***i.)* Self-aligning roller bearings:**

These bearings are self-adjustable and it is not affected by non-centering of shaft deformation flexure of shaft and bearing block, so it can compensate the concentricity error caused by these reason.




***ii.)* Tapered roller bearings:**

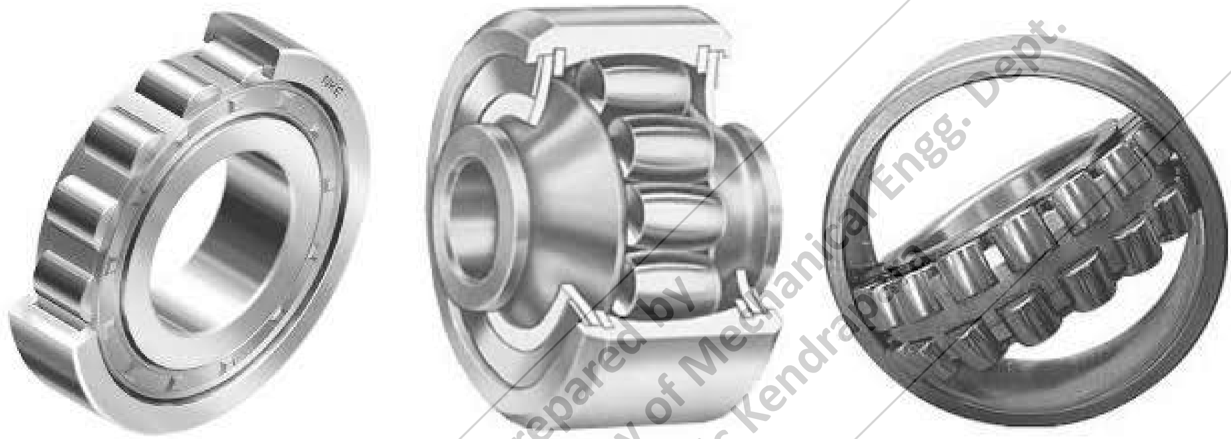
These bearings are used to take thrust only one direction. For opposing thrust, the bearings must be mounted in opposite pair. Tapered roller bearings are mainly used for high axial thrust loads.

***iii.)*Needle roller bearings:**

These types of bearings have very small roller diameter. Rolling element is called needle roller. Needle roller bearings are used where the outside diameter of bearing restricted due to limited bearing space in the housing. The needles fitted in a circular cage which is push-fit in its housing.

***iv.)* Cylindrical roller bearings:**

These bearings are used in such places, where more load is to be bear. These are made of hollow cylindrical roller.





***v.)* Barrel roller bearings:**

The roller fitted in this type of bearing has barrel shape. Its diameter up to full length is uneven. These bearings are self-alignment type bearing, so due to this reason, it has no difficulty to maintain shaft in straight line.

***vi.)* Spherical roller bearings:**

These bearings are used in such places, where chance of angular condition in shaft may be possible. In these bearings, rollers are fitted in ball cage in such a way that in angular condition, in inner and outer race, the energy transfer can be possible in full capacity.

**Advantages of Friction bearing:**

- Special shielded bearing does not required lubrication again.
- It is easy to replace.
- It has very long life and has very less friction.
- It easily operates on high speed and required less lubrication.

**Disadvantages of Friction bearing:**

- Initial cost is usually high.
- Greater diameter space required for comparable shaft diameter.
- Dirt, metal chips and so on, entering the bearings can limit their life causing early failure.
- Lesser capacity to withstand shock.

**B. According to the load:**

**1. Radial or journal bearing:**

In this type of bearing, the loading is at right angles to the bearing axis, such that bearing is installed perpendicular to the axial line of the shaft.


**2. Thrust bearing:**

In these types of bearings, the loading is parallel to the bearings axis and collar is used to rotate the shaft at one position.

**3. Pivot bearing:**

These bearings are used to the give support to the shaft in stationary position wherever bearings are in parallel to the axis of the shaft and one edge of the shaft is in the inside of the bearing. These bearings are also known as foot bearing.



**4. Slipper bearing:**

These bearings provides supports to the moving part or part in the straight line such as cross head which in used in steam engine.

**C. According to the load:**

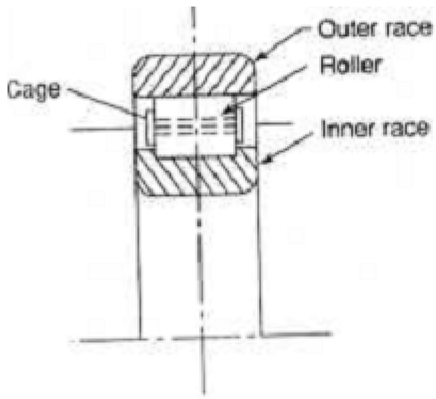
**1. Flat shape bearing:**

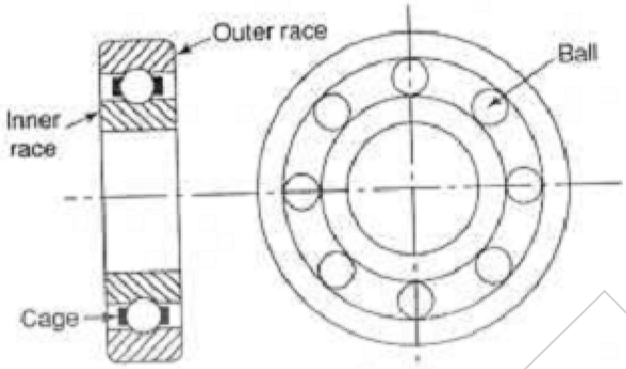
These types of bearings are flat in shapes which provide support to the sliding parts. Therefore, these bearings are also called slide bearing or guide bearing such as the movement of carriage on lathe bed.


**2. Round or cylindrical bearing:**

These bearings are in round shape which provides support to the moving parts as in case of solid bearing, ball bearing or roller bearing.

**<u>Ball bearing and Roller bearing</u>**

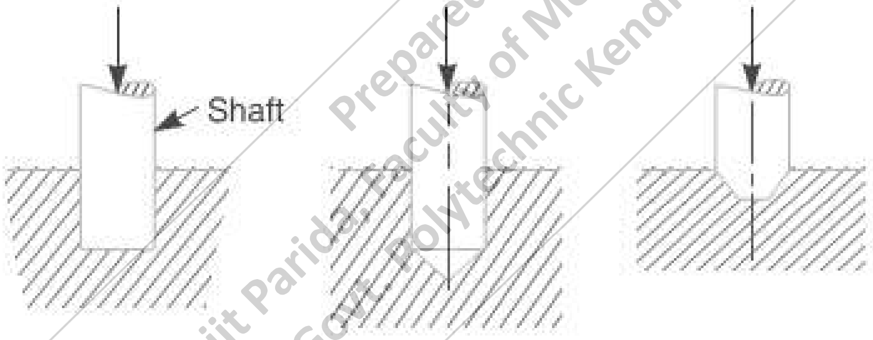



- Inner race is moving
- Outer race is stationary
- Anti friction element i.e. ball/roller/needle
- Cage which separates the balls or rollers from each other.

**Pivot Bearing:**

The rotating shafts are frequently subjected to axial thrust. The propeller shafts of ships, the shafts of steam turbines, and vertical machine shafts are examples of shafts which carry an axial thrust. The bearing surfaces placed at the end of a shaft to take the axial thrust are known as *pivots*.




(*a*) Flat pivot. (*b*) Conical pivot. (*c*) Truncated pivot.

A little consideration will show that in a new bearing, the contact between the shaft and bearing may be good over the whole surface. In other words, we can say that the pressure over the rubbing surfaces is uniformly distributed. But when the bearing becomes old, all parts of the rubbing surface will not move with the same velocity, because the velocity of rubbing surface increases with the distance from the axis of the bearing. This means that wear may be different at different radii and this causes to alter the distribution of pressure. Hence, in the study of friction of bearings, it is assumed that

*1.* The pressure is uniformly distributed throughout the bearing surface, and
*2.* The wear is uniform throughout the bearing surface.

**Flat Pivot Bearing:**

When a vertical shaft rotates in a flat pivot bearing (known as foot step bearing), as shown in Fig. 10.17, the sliding friction will be along the surface of contact between the shaft and the bearing.

Let W = Load transmitted over the bearing surface,

R = Radius of bearing surface,

p = Intensity of pressure per unit area of bearing surface between rubbing surfaces, and

μ = Coefficient of friction.

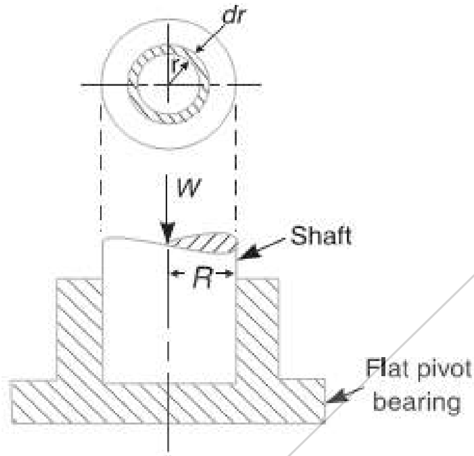



**Fig. 10.17.** Flat pivot or footstep bearing.

We will consider the following two cases:

***1. When there is a uniform pressure:***

When the pressure is uniformly distributed over the bearing area, then

$$p = \frac{W}{\pi R^2}$$

Consider a ring of radius $r$ and thickness $dr$ of the bearing area.

$\therefore$ Area of bearing surface, $A = 2\pi r.dr$

Load transmitted to the ring,

$$\delta W = p \times A = p \times 2\pi r.dr \quad ...(i)$$

Frictional resistance to sliding on the ring acting tangentially at radius $r$,

$$F_r = \mu.\delta W = \mu p \times 2\pi r.dr = 2\pi \mu.p.r.dr$$

$\therefore$ Frictional torque on the ring,

$$T_r = F_r \times r = 2\pi \mu p\, r.dr \times r = 2\pi \mu p r^2 dr \quad ...(ii)$$

Integrating this equation within the limits from 0 to $R$ for the total frictional torque on the pivot bearing.

$$\therefore \text{ Total frictional torque, } T = \int_0^R 2\pi\mu p r^2\, dr = 2\pi\mu p \int_0^R r^2\, dr$$

$$= 2\pi\mu p \left[\frac{r^3}{3}\right]_0^R = 2\pi\mu p \times \frac{R^3}{3} = \frac{2}{3} \times \pi\mu.p.R^3$$

$$= \frac{2}{3} \times \pi\mu \times \frac{W}{\pi R^2} \times R^3 = \frac{2}{3} \times \mu.W.R \quad ...\left(\because\ p = \frac{W}{\pi R^2}\right)$$

When the shaft rotates at ω rad/s, then power lost in friction,

$$P = T.\omega = T \times 2\pi N/60 \quad ...(\because\ \omega = 2\pi N/60)$$

where $N$ = Speed of shaft in r.p.m.

***2. When there is a uniform wear:***

We have already discussed that the rate of wear depends upon the intensity of pressure ($p$) and the velocity of rubbing surfaces ($v$). It is assumed that the rate of wear is proportional to the product of intensity of pressure and the velocity of rubbing surfaces (*i.e.* $p.v.$). Since the velocity of rubbing surfaces increases with the distance (*i.e.* radius $r$) from the axis of the bearing, therefore for uniform wear

$$p.r = C \text{ (a constant)} \quad \text{or} \quad p = C/r$$

and the load transmitted to the ring,

$$\delta W = p \times 2\pi r.dr \qquad \text{...[From equation } (i)]$$

$$= \frac{C}{r} \times 2\pi r.dr = 2\pi C.dr$$

$\therefore$ Total load transmitted to the bearing

$$W = \int_0^R 2\pi\, C.dr = 2\pi\, C[r]_0^R = 2\pi C.R \quad \text{or} \quad C = \frac{W}{2\pi R}$$

We know that frictional torque acting on the ring,

$$T_r = 2\pi\mu\, p\, r^2\, dr = 2\pi\mu \times \frac{C}{r} \times r^2\, dr \qquad \text{...}\left(\because\ p = \frac{C}{r}\right)$$

$$= 2\pi\, \mu.C.r\, dr \qquad \text{...}(iii)$$

$\therefore$ Total frictional torque on the bearing,

$$T = \int_0^R 2\pi\, \mu.C.r.dr = 2\pi\mu.C\left[\frac{r^2}{2}\right]_0^R$$

$$= 2\pi\mu.C \times \frac{R^2}{2} = \pi\mu.C.R^2$$

$$= \pi\mu \times \frac{W}{2\pi R} \times R^2 = \frac{1}{2} \times \mu.W.R \qquad \text{...}\left(\because\ C = \frac{W}{2\pi R}\right)$$

**Example 10.16.** *A vertical shaft 150 mm in diameter rotating at 100 r.p.m. rests on a flat end footstep bearing. The shaft carries a vertical load of 20 kN. Assuming uniform pressure distribution and coefficient of friction equal to 0.05, estimate power lost in friction.*

**Solution.** Given : $D = 150$ mm or $R = 75$ mm $= 0.075$ m ; $N = 100$ r.p.m or $\omega = 2\pi \times 100/60 = 10.47$ rad/s ; $W = 20$ kN $= 20 \times 10^3$ N ; $\mu = 0.05$

We know that for uniform pressure distribution, the total frictional torque,

$$T = \frac{2}{3} \times \mu.W.R = \frac{2}{3} \times 0.05 \times 20 \times 10^3 \times 0.075 = 50 \text{ N-m}$$

$\therefore$ Power lost in friction,

$$P = T.\omega = 50 \times 10.47 = 523.5 \text{ W} \quad \textbf{Ans.}$$

## Conical Pivot Bearing:

The conical pivot bearing supporting a shaft carrying a load W is shown in Fig. 10.18.

Let $P_n$ = Intensity of pressure normal to the cone,

$\alpha$ = Semi angle of the cone,

$\mu$ = Coefficient of friction between the shaft and the bearing, and

R = Radius of the shaft.

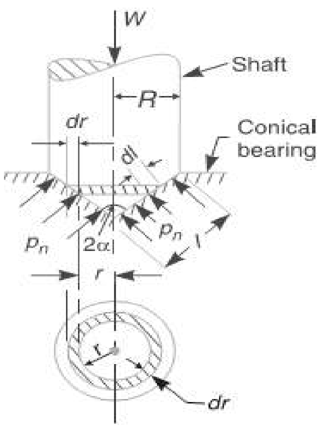



**Fig. 10.18.**
Conical pivot bearing.

Consider a small ring of radius r and thickness dr. Let dl is the length of ring along the cone, such that

$$dl = dr \text{ cosec } \alpha$$

∴ Area of the ring, $A = 2\pi r.dl = 2\pi r.dr \text{ cosec } \alpha$ ...($\because$ dl = dr cosec α)

***1. When there is a uniform pressure:***

We know that normal load acting on the ring,

$$\delta W_n = \textit{Normal pressure} \times \textit{Area}$$

$$= p_n \times 2\pi r.dr \text{ cosec } \alpha$$

and vertical load acting on the ring,

$$^*\delta W = \text{Vertical component of } \delta W_n = \delta W_n.\sin\alpha$$

$$= p_n \times 2\pi r.dr \text{ cosec } \alpha.\ \sin\alpha = p_n \times 2\pi\ r.dr$$

∴ Total vertical load transmitted to the bearing,

$$W = \int_0^R p_n \times 2\pi r.dr = 2\pi\, p_n \left[\frac{r^2}{2}\right]_0^R = 2\pi\, p_n \times \frac{R^2}{2} = \pi R^2.p_n$$

or $$p_n = W/\pi R^2$$

We know that frictional force on the ring acting tangentially at radius $r$,

$$F_r = \mu.\delta W_n = \mu.p_n.2\pi r.dr \text{ cosec } \alpha = 2\pi\mu.p_n.\text{cosec } \alpha.r.dr$$

and frictional torque acting on the ring,

$$T_r = F_r \times r = 2\pi\mu.p_n.\text{cosec } \alpha.r.dr \times r = 2\pi\mu.p_n \text{ cosec}\alpha.r^2.dr$$

Integrating the expression within the limits from 0 to $R$ for the total frictional torque on the conical pivot bearing.

∴ Total frictional torque,

$$T = \int_0^R 2\pi\mu.p_n \text{ cosec } \alpha.r^2\, dr = 2\pi\mu.p_n.\text{cosec } \alpha \left[\frac{r^3}{3}\right]_0^R$$

$$= 2\pi\mu.p_n.\text{cosec } \alpha \times \frac{R^3}{3} = \frac{2\pi R^3}{3} \times \mu.p_n.\text{cosec } \alpha \quad ...(i)$$

Substituting the value of $p_n$ in equation ($i$),

$$T = \frac{2\pi R^3}{3} \times \pi \times \frac{W}{\pi R^2} \times \text{cosec } \alpha = \frac{2}{3} \times \mu W.R.\text{ cosec } \alpha$$

**Note :** If slant length ($l$) of the cone is known, then

$$T = \frac{2}{3} \times \mu.W\,l \quad ...(\because\ l = R \text{ cosec } \alpha)$$

***2. When there is a uniform wear:***

In Fig. 10.18, let $p_r$ be the normal intensity of pressure at a distance r from the central axis. We know that, in case of uniform wear, the intensity of pressure varies inversely with the distance.

∴ $$p_r.r = C \text{ (a constant) or } p_r = C/r,$$

and the load transmitted to the ring,

$$\delta W = p_r \times 2\pi r.dr = \frac{C}{r} \times 2\pi r.dr = 2\pi C.dr$$

∴ Total load transmitted to the bearing,

$$W = \int_0^R 2\pi C.dr = 2\pi C\,[r]_0^R = 2\pi C.R \quad \text{or} \quad C = \frac{W}{2\pi R}$$

We know that frictional torque acting on the ring,

$$T_r = 2\pi\mu.p_r.\text{cosec}\,\alpha.r^2.dr = 2\pi\mu \times \frac{C}{r} \times \text{cosec}\,\alpha.r^2.dr$$

$$= 2\pi\mu.C.\text{cosec}\,\alpha.r.dr$$

∴ Total frictional torque acting on the bearing,

$$T = \int_0^R 2\pi\,\mu.C.\text{cosec}\,\alpha.r.dr = 2\pi\mu.C.\text{cosec}\,\alpha\left[\frac{r^2}{2}\right]_0^R$$

$$= 2\pi\,\mu.C.\text{cosec}\,\alpha \times \frac{R^2}{2} = \pi\mu.C.\text{cosec}\,\alpha.R^2$$

Substituting the value of $C$, we have

$$T = \pi\mu \times \frac{W}{2\pi R} \times \text{cosec}\,\alpha.R^2 = \frac{1}{2} \times \mu.W.R\ \text{cosec}\,\alpha = \frac{1}{2} \times \mu.W.l$$

**Example 10.17.** *A conical pivot supports a load of 20 kN, the cone angle is 120° and the intensity of normal pressure is not to exceed 0.3 N/mm². The external diameter is twice the internal diameter. Find the outer and inner radii of the bearing surface. If the shaft rotates at 200 r.p.m. and the coefficient of friction is 0.1, find the power absorbed in friction. Assume uniform pressure.*

**Solution.** Given : $W$ = 20 kN = 20 × 10³ N ; 2 α = 120° or α = 60° ; $p_n$ = 0.3 N/mm² ; $N$ = 200 r.p.m. or ω = 2 π × 200/60 = 20.95 rad/s ; μ = 0.1

***Outer and inner radii of the bearing surface***

Let $r_1$ and $r_2$ = Outer and inner radii of the bearing surface, in mm.

Since the external diameter is twice the internal diameter, therefore

$$r_1 = 2\,r_2$$

We know that intensity of normal pressure ($p_n$),

$$0.3 = \frac{W}{\pi[(r_1)^2 - (r_2)^2]} = \frac{20 \times 10^3}{\pi[(2r_2)^2 - (r_2)^2]} = \frac{2.12 \times 10^3}{(r_2)^2}$$

∴ $(r_2)^2 = 2.12 \times 10^3 / 0.3 = 7.07 \times 10^3$ or $r_2 = 84$ mm **Ans.**

and $r_1 = 2\,r_2 = 2 \times 84 = 168$ mm **Ans.**

***Power absorbed in friction***

We know that total frictional torque (assuming uniform pressure),

$$T = \frac{2}{3} \times \mu.W.\text{cosec}\,\alpha\left[\frac{(r_1)^3 - (r_2)^3}{(r_1)^2 - (r_2)^2}\right]$$

$$= \frac{2}{3} \times 0.1 \times 20 \times 10^3 \times \text{cosec}\,60° = \left[\frac{(168)^3 - (84)^3}{(168)^2 - (84)^2}\right] \text{N-mm}$$

$$= 301760 \text{ N-mm} = 301.76 \text{ N-m}$$

∴ Power absorbed in friction,

$$P = T.\omega = 301.76 \times 20.95 = 6322 \text{ W} = 6.322 \text{ kW} \quad \textbf{Ans.}$$

**Example 10.18.** *A conical pivot bearing supports a vertical shaft of 200 mm diameter. It is subjected to a load of 30 kN. The angle of the cone is 120° and the coefficient of friction is 0.025. Find the power lost in friction when the speed is 140 r.p.m., assuming 1. uniform pressure ; and 2. uniform wear.*

**Solution.** Given : $D = 200$ mm or $R = 100$ mm $= 0.1$ m ; $W = 30$ kN $= 30 \times 10^3$ N ; $2\alpha = 120°$ or $\alpha = 60°$ ; $\mu = 0.025$ ; $N = 140$ r.p.m. or $\omega = 2\pi \times 140/160 = 14.66$ rad/s

**1.** ***Power lost in friction assuming uniform pressure***

We know that total frictional torque,

$$T = \frac{2}{3} \times \mu W.R. \operatorname{cosec} \alpha$$

$$= \frac{2}{3} \times 0.025 \times 30 \times 10^3 \times 0.1 \times \operatorname{cosec} 60° = 57.7 \text{ N-m}$$

$\therefore$ Power lost in friction,

$$P = T.\omega = 57.7 \times 14.66 = 846 \text{ W } \textbf{Ans.}$$

**2.** ***Power lost in friction assuming uniform wear***

We know that total frictional torque,

$$T = \frac{1}{2} \times \mu W.R. \operatorname{cosec} \alpha$$

$$= \frac{1}{2} \times 0.025 \times 30 \times 10^3 \times 0.1 \times \operatorname{cosec} 60° = 43.3 \text{ N-m}$$

$\therefore$ Power lost in friction, $P = T.\omega = 43.3 \times 14.66 = 634.8$ W **Ans.**



**Flat Collar Bearing:**

We have already discussed that collar bearings are used to take the axial thrust of the rotating shafts. There may be a single collar or multiple collar bearings as shown in Fig. 10.20 (a) and (b) respectively. The collar bearings are also known as thrust bearings.

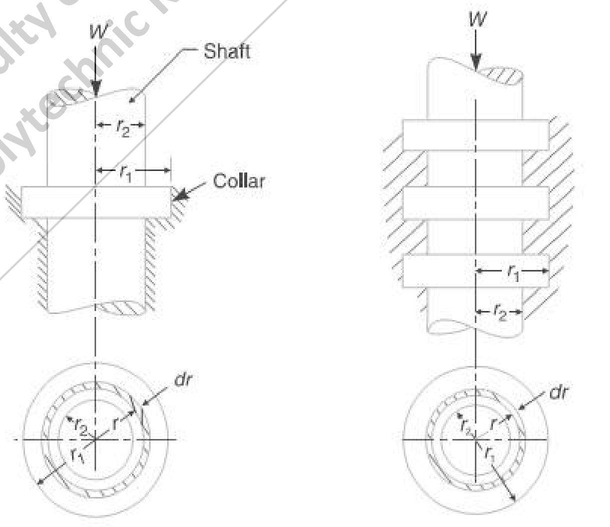
Consider a single flat collar bearing supporting a shaft as shown in Fig. 10.20 (a).

Let $r_1$ = External radius of the collar, and

$r_2$ = Internal radius of the collar.

$\therefore$ Area of the bearing surface,

$$A = \pi [(r_1)^2 - (r_2)^2]$$




(*a*) Single collar bearing (*b*) Multiple collar bearing.

**Fig. 10.20.** Flat collar bearings.

***1. When there is a uniform pressure:***

When the pressure is uniformly distributed over the bearing surface, then the intensity of pressure,

$$p = \frac{W}{A} = \frac{W}{\pi[(r_1)^2 - (r_2)^2]} \qquad \textbf{...(i)}$$

We have seen in Art. 10.25, that the frictional torque on the ring of radius $r$ and thickness $dr$,

$$T_r = 2\pi\mu.p.r^2.dr$$

Integrating this equation within the limits from $r_2$ to $r_1$ for the total frictional torque on the collar.

∴ Total frictional torque,

$$T = \int_{r_2}^{r_1} 2\pi\mu.p.r^2.dr = 2\pi\mu.p\left[\frac{r_3}{3}\right]_{r_2}^{r_1} = 2\pi\mu.p\left[\frac{(r_1)^3 - (r_2)^3}{3}\right]$$

Substituting the value of $p$ from equation (*i*),

$$T = 2\pi\mu \times \frac{W}{\pi[(r_1)^2 - (r_2)^2]}\left[\frac{(r_1)^3 - (r_2)^3}{3}\right]$$

$$= \frac{2}{3} \times \mu.W\left[\frac{(r_1)^3 - (r_2)^3}{(r_1)^2 - (r_2)^2}\right]$$

**Notes: 1.** In order to increase the amount of rubbing surfaces so as to reduce the intensity of pressure, it is better to use two or more collars, as shown in Fig. 10.20 (*b*), rather than one larger collar.

**2.** In case of a multi-collared bearings with, say $n$ collars, the intensity of the uniform pressure,

$$p = \frac{\text{Load}}{\text{No. of collars} \times \text{Bearing area of one collar}} = \frac{W}{n\pi[(r_1)^2 - (r_2)^2]}$$

**3.** The total torque transmitted in a multi collared shaft remains constant *i.e.*

$$T = \frac{2}{3} \times \mu.W\left[\frac{(r_1)^3 - (r_2)^3}{(r_1)^2 - (r_2)^2}\right]$$



**2. *Considering unifrom wear***

We have seen in Art. 10.25 that the load transmitted on the ring, considering uniform wear is,

$$\delta W = p_r.2\pi r.dr = \frac{C}{r} \times 2\pi r.dr = 2\pi C.dr$$

∴ Total load transmitted to the collar,

$$W = \int_{r_2}^{r_1} 2\pi C.dr = 2\pi C[r]_{r_2}^{r_1} = 2\pi C(r_1 - r_2)$$

or

$$C = \frac{W}{2\pi(r_1 - r_2)} \quad ...(ii)$$

We also know that frictional torque on the ring,

$$T_r = \mu.\delta W.r = \mu \times 2\pi C.dr.r = 2\pi\mu.C.r.dr$$

∴ Total frictional torque on the bearing,

$$T = \int_{r_2}^{r_1} 2\pi\mu C.r.dr = 2\pi\mu.C\left[\frac{r^2}{2}\right]_{r_2}^{r_1} = 2\pi\mu.C\left[\frac{(r_1)^2 - (r_2)^2}{2}\right]$$

$$= \pi\mu.C[(r_1)^2 - (r_2)^2]$$

Substituting the value of $C$ from equation (*ii*),

$$T = \pi\mu \times \frac{W}{2\pi(r_1 - r_2)}[(r_1)^2 - (r_2)^2] = \frac{1}{2} \times \mu.W(r_1 + r_2)$$

**Example 10.19.** *A thrust shaft of a ship has 6 collars of 600 mm external diameter and 300 mm internal diameter. The total thrust from the propeller is 100 kN. If the coefficient of friction is 0.12 and speed of the engine 90 r.p.m., find the power absorbed in friction at the thrust block, assuming* **1.** *uniform pressure ; and* **2.** *uniform wear.*

Ship propeller.

**Solution.** Given : $n = 6$ ; $d_1 = 600$ mm or $r_1 = 300$ mm ; $d_2 = 300$ mm or $r_2 = 150$ mm ; $W = 100$ kN $= 100 \times 10^3$ N ; $\mu = 0.12$ ; $N = 90$ r.p.m. or $\omega = 2\pi \times 90/60 = 9.426$ rad/s

**1.** ***Power absorbed in friction, assuming uniform pressure***

We know that total frictional torque transmitted,

$$T = \frac{2}{3} \times \mu . W \left[ \frac{(r_1)^3 - (r_2)^3}{(r_1)^2 - (r_2)^2} \right]$$

$$= \frac{2}{3} \times 0.12 \times 100 \times 10^3 \left[ \frac{(300)^3 - (150)^3}{(300)^2 - (150)^2} \right] = 2800 \times 10^3 \text{ N-mm}$$

$$= 2800 \text{ N-m}$$

$\therefore$ Power absorbed in friction,

$$P = T.\omega = 2800 \times 9.426 = 26\,400 \text{ W} = 26.4 \text{ kW } \textbf{Ans.}$$

**2.** ***Power absorbed in friction assuming uniform wear***

We know that total frictional torque transmitted,

$$T = \frac{1}{2} \times \mu . W (r_1 + r_2) = \frac{1}{2} \times 0.12 \times 100 \times 10^3 \; (300 + 150) \text{ N-mm}$$

$$= 2700 \times 10^3 \text{ N-mm} = 2700 \text{ N-m}$$

$\therefore$ Power absorbed in friction,

$$P = T.\omega = 2700 \times 9.426 = 25\,450 \text{ W} = 25.45 \text{ kW } \textbf{Ans.}$$

**Example 10.20.** *A shaft has a number of a collars integral with it. The external diameter of the collars is 400 mm and the shaft diemater is 250 mm. If the intensity of pressure is 0.35 N/mm² (uniform) and the coefficient of friction is 0.05, estimate :* **1.** *power absorbed when the shaft runs at 105 r.p.m. carrying a load of 150 kN ; and* **2.** *number of collars required.*

**Solution.** Given : $d_1 = 400$ mm or $r_1 = 200$ mm ; $d_2 = 250$ mm or $r_2 = 125$ mm ; $p = 0.35$ N/mm² ; $\mu = 0.05$ ; $N = 105$ r.p.m or $\omega = 2\pi \times 105/60 = 11$ rad/s ; $W = 150$ kN $= 150 \times 10^3$ N

**1.** ***Power absorbed***

We know that for uniform pressure, total frictional torque transmitted,

$$T = \frac{2}{3} \times \mu W \left[ \frac{(r_1)^3 - (r_2)^3}{(r_1)^2 - (r_2)^2} \right] = \frac{2}{3} \times 0.05 \times 150 \times 10^3 \left[ \frac{(200)^3 - (125)^3}{(200)^2 - (125)^2} \right] \text{N-mm}$$

$$= 5000 \times 248 = 1240 \times 10^3 \text{ N-mm} = 1240 \text{ N-m}$$

$\therefore$ Power absorbed,

$$P = T.\omega = 1240 \times 11 = 13640 \text{ W} = 13.64 \text{ kW } \textbf{Ans.}$$

**2.** ***Number of collars required***

Let $n$ = Number of collars required.

We know that the intensity of uniform pressure ($p$),

$$0.35 = \frac{W}{n.\pi[(r_1)^2 - (r_2)^2]} = \frac{150 \times 10^3}{n.\pi[(200)^2 - (125)^2]} = \frac{1.96}{n}$$

$\therefore$ $n = 1.96/0.35 = 5.6$ say 6 **Ans.**

## Friction Clutches:

A friction clutch has its principal application in the transmission of power of shafts and machines which must be started and stopped frequently. The force of friction is used to start the driven shaft from rest and gradually brings it up to the proper speed without excessive slipping of the friction surfaces. In automobiles, friction clutch is used to connect the engine to the driven shaft. It may be noted that

- The contact surfaces should develop a frictional force that may pick up and hold the load with reasonably low pressure between the contact surfaces.
- The heat of friction should be rapidly dissipated and tendency to grab should be at a minimum.
- The surfaces should be backed by a material stiff enough to ensure a reasonably uniform distribution of pressure.

The friction clutches of the following types are important from the subject point of view:

**a)** Disc or plate clutches (single disc or multiple disc clutch),
**b)** Cone clutches, and
**c)** Centrifugal clutches.

### *1. Single Disc or Plate Clutch:*

A single disc or plate clutch, as shown in Fig. 10.21, consists of a clutch plate whose both sides are faced with a friction material (usually of Ferrodo). It is mounted on the hub which is free to move axially along the splines of the driven shaft. The pressure plate is mounted inside the clutch body which is bolted to the flywheel. Both the pressure plate and the flywheel rotate with the engine crankshaft or the driving shaft. The pressure plate pushes the clutch plate towards the flywheel by a set of strong springs which are arranged radially inside the body. The three levers (also known as release levers or fingers) are carried on pivots suspended from the case of the body. These are arranged in such a manner so that the pressure plate moves away from the flywheel by the inward movement of a thrust bearing. The bearing is mounted upon a forked shaft and moves forward when the clutch pedal is pressed. When the clutch pedal is pressed down, its linkage forces the thrust release bearing to move in towards the flywheel and pressing the longer ends of the levers inward. The levers are forced to turn on their suspended pivot and the pressure plate moves away from the flywheel by the knife edges, thereby compressing the clutch springs. This action removes the pressure from the clutch plate and thus moves back from the flywheel and the driven shaft becomes stationary. On the other hand, when the foot is taken off from the clutch pedal, the thrust bearing moves back by the levers. This allows the springs to extend and thus the pressure plate pushes the clutch plate back towards the flywheel.

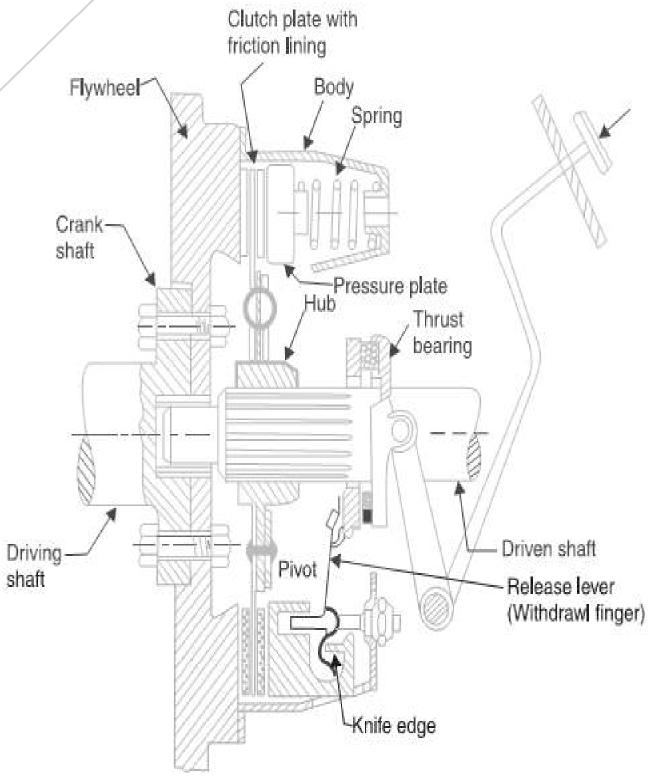



Fig. 10.21. Single disc or plate clutch.

The axial pressure exerted by the spring provides a frictional force in the circumferential direction when the relative motion between the driving and driven members tends to take place. If the torque due to this frictional force exceeds the torque to be transmitted, then no slipping takes place and the power is transmitted from the driving shaft to the driven shaft.

Now consider two friction surfaces, maintained in contact by an axial thrust W, as shown in Fig. 10.22 (a).

Let T = Torque transmitted by the clutch,

p = Intensity of axial pressure with which the contact surfaces are held together,

$r_1$ and $r_2$ = External and internal radii of friction faces, and

μ = Coefficient of friction.

Consider an elementary ring of radius r and thickness *dr* as shown in Fig. 10.22 (b).

We know that area of contact surface or friction surface,

$$= 2\pi r.dr$$

∴ Normal or axial force on the ring,

$$\delta W = Pressure \times Area = p \times 2\pi r.dr$$

and the frictional force on the ring acting tangentially at radius r,

$$F_r = \mu.\delta W = \mu.p \times 2\pi r.dr$$

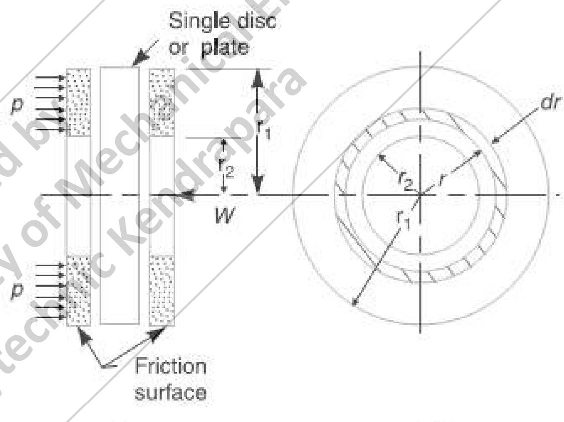



Fig. 10.22. Forces on a single disc or plate clutch.

∴ Frictional torque acting on the ring,

$$T_r = F_r \times r = \mu.p \times 2\pi r.dr \times r = 2\pi \times \mu.p.r^2\, dr$$

**1. *Considering uniform pressure***

When the pressure is uniformly distributed over the entire area of the friction face, then the intensity of pressure,

$$p = \frac{W}{\pi[(r_1)^2 - (r_2)^2]} \qquad \ldots(i)$$

where $W$ = Axial thrust with which the contact or friction surfaces are held together.

We have discussed above that the frictional torque on the elementary ring of radius $r$ and thickness $dr$ is

$$T_r = 2\pi\mu.p.r^2\, dr$$

Integrating this equation within the limits from $r_2$ to $r_1$ for the total frictional torque.

∴ Total frictional torque acting on the friction surface or on the clutch,

$$T = \int_{r_1}^{r_2} 2\pi\mu.p.r^2.dr = 2\pi\mu p\left[\frac{r^3}{3}\right]_{r_2}^{r_1} = 2\pi\mu p\left[\frac{(r_1)^3 - (r_2)^3}{3}\right]$$

Substituting the value of $p$ from equation (*i*),

$$T = 2\pi\mu \times \frac{W}{\pi[(r_1)^2 - (r_2)^2]} \times \frac{(r_1)^3 - (r_2)^3}{3}$$

$$= \frac{2}{3} \times \mu W \left[\frac{(r_1)^3 - (r_2)^3}{(r_1)^2 - (r_2)^2}\right] = \mu.W.R$$

where $R$ = Mean radius of friction surface

$$= \frac{2}{3}\left[\frac{(r_1)^3 - (r_2)^3}{(r_1)^2 - (r_2)^2}\right]$$

**2.** ***Considering uniform wear***

In Fig. 10.22, let $p$ be the normal intensity of pressure at a distance $r$ from the axis of the clutch. Since the intensity of pressure varies inversely with the distance, therefore

$$p.r. = C \text{ (a constant)} \quad \text{or} \quad p = C/r \qquad \text{...}(i)$$

and the normal force on the ring,

$$\delta W = p.2\pi r.dr = \frac{C}{r} \times 2\pi C.dr = 2\pi C.dr$$

∴ Total force acting on the friction surface,

$$W = \int_{r_2}^{r_1} 2\pi C\, dr = 2\pi C[r]_{r_2}^{r_1} = 2\pi C(r_1 - r_2)$$

or

$$C = \frac{W}{2\pi(r_1 - r_2)}$$

We know that the frictional torque acting on the ring,

$$T_r = 2\pi\mu.p\,r^2.dr = 2\pi\mu \times \frac{C}{r} \times r^2.dr = 2\pi\mu.C.r.dr$$

$$\text{...}(\because\ p = C/r)$$

∴ Total frictional torque on the friction surface,

$$T = \int_{r_2}^{r_1} 2\pi\mu.C.r.dr = 2\pi\mu.C\left[\frac{r^2}{2}\right]_{r_2}^{r_1} = 2\pi\mu.C\left[\frac{(r_1)^2 - (r_2)^2}{2}\right]$$

$$= \pi\mu.C[(r_1)^2 - (r_2)^2] = \pi\mu \times \frac{W}{2\pi(r_1 - r_2)}[(r_1)^2 - (r_2)^2]$$

$$= \frac{1}{2} \times \mu.W(r_1 + r_2) = \mu.W.R$$

where $R$ = Mean radius of the friction surface $= \dfrac{r_1 + r_2}{2}$

**Notes : 1.** In general, total frictional torque acting on the friction surface (or on the clutch) is given by

$$T = n.\mu.W.R$$

where $n$ = Number of pairs of friction or contact surfaces, and

$R$ = Mean radius of friction surface

$$= \frac{2}{3}\left[\frac{(r_1)^3 - (r_2)^3}{(r_1)^2 - (r_2)^2}\right] \qquad \text{...(For uniform pressure)}$$

$$= \frac{r_1 + r_2}{2} \qquad \text{...(For uniform wear)}$$

**2.** For a single disc or plate clutch, normally both sides of the disc are effective. Therefore, a single disc clutch has two pairs of surfaces in contact, *i.e.* $n = 2$.

**3.** Since the intensity of pressure is maximum at the inner radius ($r_2$) of the friction or contact surface, therefore equation (*i*) may be written as

$$p_{max} \times r_2 = C \quad \text{or} \quad p_{max} = C/r_2$$

**4.** Since the intensity of pressure is minimum at the outer radius ($r_1$) of the friction or contact surface, therefore equation (*i*) may be written as

$$p_{min} \times r_1 = C \quad \text{or} \quad p_{min} = C/r_1$$

5. The average pressure ($p_{av}$) on the friction or contact surface is given by

$$p_{av} = \frac{\text{Total force on friction surface}}{\text{Cross-sectional area of friction surface}} = \frac{W}{\pi[(r_1)^2 - (r_2)^2]}$$

6. In case of a new clutch, the intensity of pressure is approximately uniform but in an old clutch the uniform wear theory is more approximate.

7. The uniform pressure theory gives a higher frictional torque than the uniform wear theory. Therefore in case of friction clutches, uniform wear should be considered, unless otherwise stated.

**Example 10.22.** *Determine the maximum, minimum and average pressure in plate clutch when the axial force is 4 kN. The inside radius of the contact surface is 50 mm and the outside radius is 100 mm. Assume uniform wear.*

**Solution.** Given : $W = 4$ kN $= 4 \times 10^3$ N ; $r_2 = 50$ mm ; $r_1 = 100$ mm

***Maximum pressure***

Let $p_{max}$ = Maximum pressure.

Since the intensity of pressure is maximum at the inner radius ($r_2$), therefore

$$p_{max} \times r_2 = C \quad \text{or} \quad C = 50\, p_{max}$$

We know that the total force on the contact surface ($W$),

$$4 \times 10^3 = 2\,\pi\, C\,(r_1 - r_2) = 2\,\pi \times 50\, p_{max}\,(100 - 50) = 15\ 710\, p_{max}$$

$$\therefore \quad p_{max} = 4 \times 10^3/15\ 710 = 0.2546 \text{ N/mm}^2 \quad \textbf{Ans.}$$

***Minimum pressure***

Let $p_{min}$ = Minimum pressure.

Since the intensity of pressure is minimum at the outer radius ($r_1$), therefore

$$p_{min} \times r_1 = C \quad \text{or} \quad C = 100\, p_{min}$$

We know that the total force on the contact surface ($W$),

$$4 \times 10^3 = 2\,\pi\, C\,(r_1 - r_2) = 2\pi \times 100\, p_{min}\,(100 - 50) = 31\ 420\, p_{min}$$

$$\therefore \quad p_{min} = 4 \times 10^3/31\ 420 = 0.1273 \text{ N/mm}^2 \quad \textbf{Ans.}$$

***Average pressure***

We know that average pressure,

$$p_{av} = \frac{\text{Total normal force on contact surface}}{\text{Cross-sectional area of contact surfaces}}$$

$$= \frac{W}{\pi[(r_1)^2 - (r_2)^2]} = \frac{4 \times 10^3}{\pi[(100)^2 - (50)^2]} = 0.17 \text{ N/mm}^2 \quad \textbf{Ans.}$$

**Example 10.23.** *A single plate clutch, with both sides effective, has outer and inner diameters 300 mm and 200 mm respectively. The maximum intensity of pressure at any point in the contact surface is not to exceed 0.1 N/mm². If the coefficient of friction is 0.3, determine the power transmitted by a clutch at a speed 2500 r.p.m.*

**Solution.** Given : $d_1 = 300$ mm or $r_1 = 150$ mm ; $d_2 = 200$ mm or $r_2 = 100$ mm ; $p = 0.1$ N/mm² ; $\mu = 0.3$ ; $N = 2500$ r.p.m. or $\omega = 2\pi \times 2500/60 = 261.8$ rad/s

Since the intensity of pressure ($p$) is maximum at the inner radius ($r_2$), therefore for uniform wear,

$$p.r_2 = C \quad \text{or} \quad C = 0.1 \times 100 = 10 \text{ N/mm}$$

We know that the axial thrust,

$$W = 2\,\pi\, C\,(r_1 - r_2) = 2\,\pi \times 10\,(150 - 100) = 3142 \text{ N}$$

and mean radius of the friction surfaces for uniform wear,

$$R = \frac{r_1 + r_2}{2} = \frac{150 + 100}{2} = 125 \text{ mm} = 0.125 \text{ m}$$

We know that torque transmitted,

$$T = n.\mu.W.R = 2 \times 0.3 \times 3142 \times 0.125 = 235.65 \text{ N-m}$$

...($\because$ $n = 2$, for both sides of plate effective)

$\therefore$ Power transmitted by a clutch,

$$P = T.\omega = 235.65 \times 261.8 = 61\ 693 \text{ W} = 61.693 \text{ kW} \quad \textbf{Ans.}$$

### *2. Multiple Disc or Plate Clutch:*

A multiple disc clutch, as shown in Fig. 10.23, may be used when a large torque is to be transmitted. The inside discs (usually of steel) are fastened to the driven shaft to permit axial motion (except for the last disc). The outside discs (usually of bronze) are held by bolts and are fastened to the housing which is keyed to the driving shaft. The multiple disc clutches are extensively used in motor cars, machine tools etc.

Let $n_1$ = Number of discs on the driving shaft, and

$n_2$ = Number of discs on the driven shaft.

∴ Number of pairs of contact surfaces,

$$n = n_1 + n_2 - 1$$

and total frictional torque acting on the friction surfaces or on the clutch,

$$T = n.\mu.W.R$$

where $R$ = Mean radius of the friction surfaces

$$= \frac{2}{3}\left[\frac{(r_1)^3 - (r_2)^3}{(r_1)^2 - (r_2)^2}\right] \quad \text{...(For uniform pressure)}$$

$$= \frac{r_1 + r_2}{2} \quad \text{...(For uniform wear)}$$

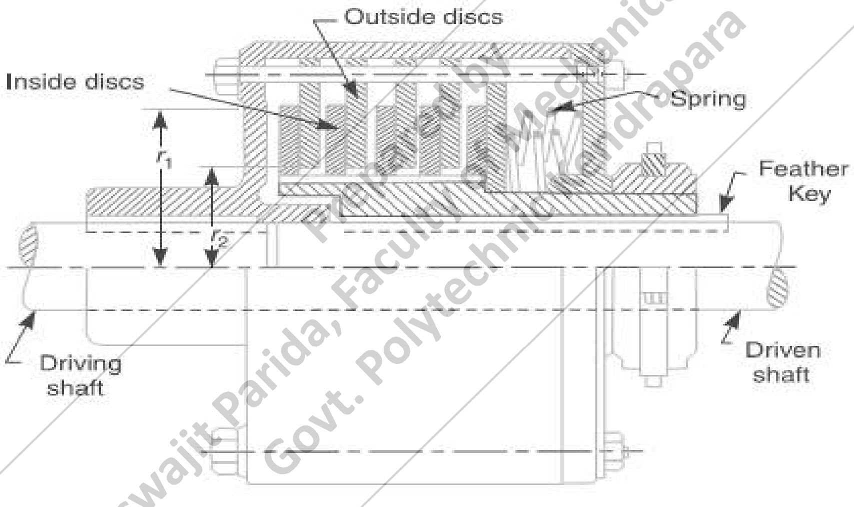



**Fig. 10.23.** Multiple disc clutch.

**Example 10.28.** *A multiple disc clutch has five plates having four pairs of active friction surfaces. If the intensity of pressure is not to exceed 0.127 N/mm², find the power transmitted at 500 r.p.m. The outer and inner radii of friction surfaces are 125 mm and 75 mm respectively. Assume uniform wear and take coefficient of friction = 0.3.*

**Solution.** Given : $n_1 + n_2 = 5$ ; $n = 4$ ; $p = 0.127$ N/mm² ; $N = 500$ r.p.m. or $\omega = 2\pi \times 500/60 = 52.4$ rad/s ; $r_1 = 125$ mm ; $r_2 = 75$ mm ; $\mu = 0.3$

Since the intensity of pressure is maximum at the inner radius $r_2$, therefore

$$p.r_2 = C \quad \text{or} \quad C = 0.127 \times 75 = 9.525 \text{ N/mm}$$

We know that axial force required to engage the clutch,

$$W = 2\pi C(r_1 - r_2) = 2\pi \times 9.525\,(125 - 75) = 2990 \text{ N}$$

and mean radius of the friction surfaces,

$$R = \frac{r_1 + r_2}{2} = \frac{125 + 75}{2} = 100 \text{ mm} = 0.1 \text{ m}$$

We know that torque transmitted,

$$T = n.\mu.W.R = 4 \times 0.3 \times 2990 \times 0.1 = 358.8 \text{ N-m}$$

∴ Power transmitted,

$$P = T.\omega = 358.8 \times 52.4 = 18\,800 \text{ W} = 18.8 \text{ kW} \quad \textbf{Ans.}$$

**Example 10.29.** *A multi-disc clutch has three discs on the driving shaft and two on the driven shaft. The outside diameter of the contact surfaces is 240 mm and inside diameter 120 mm. Assuming uniform wear and coefficient of friction as 0.3, find the maximum axial intensity of pressure between the discs for transmitting 25 kW at 1575 r.p.m.*

**Solution.** Given : $n_1 = 3$ ; $n_2 = 2$ ; $d_1 = 240$ mm or $r_1 = 120$ mm ; $d_2 = 120$ mm or $r_2 = 60$ mm ; $\mu = 0.3$ ; $P = 25$ kW $= 25 \times 10^3$ W ; $N = 1575$ r.p.m. or $\omega = 2\pi \times 1575/60 = 165$ rad/s

Let $T$ = Torque transmitted in N-m, and

$W$ = Axial force on each friction surface.

We know that the power transmitted ($P$),

$$25 \times 10^3 = T.\omega = T \times 165 \quad \text{or} \quad T = 25 \times 10^3/165 = 151.5 \text{ N-m}$$

Number of pairs of friction surfaces,

$$n = n_1 + n_2 - 1 = 3 + 2 - 1 = 4$$

and mean radius of friction surfaces for uniform wear,

$$R = \frac{r_1 + r_2}{2} = \frac{120 + 60}{2} = 90 \text{ mm} = 0.09 \text{ m}$$

We know that torque transmitted ($T$),

$$151.5 = n.\mu.W.R = 4 \times 0.3 \times W \times 0.09 = 0.108\, W$$

$$\therefore \quad W = 151.5/0.108 = 1403 \text{ N}$$

Let $p$ = Maximum axial intensity of pressure.

Since the intensity of pressure ($p$) is maximum at the inner radius ($r_2$), therefore for uniform wear

$$p.r_2 = C \quad \text{or} \quad C = p \times 60 = 60\, p \text{ N/mm}$$

We know that the axial force on each friction surface ($W$),

$$1403 = 2\pi.C(r_1 - r_2) = 2\pi \times 60\, p\,(120 - 60) = 22\,622\, p$$

$$\therefore \quad p = 1403/22\,622 = 0.062 \text{ N/mm}^2 \quad \textbf{Ans.}$$

## Friction Brakes:

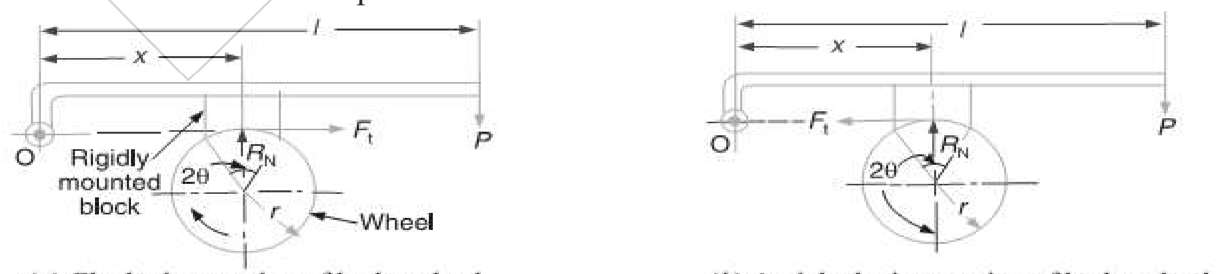
A brake is a device by means of which artificial frictional resistance is applied to a moving machine member, in order to retard or stop the motion of a machine.

## Types of Brakes:

a) Hydraulic brakes e.g. pumps or hydrodynamic brake and fluid agitator,
b) Electric brakes e.g. generators and eddy current brakes, and
c) Mechanical brakes.

## Single Block or Shoe Brake:

A single block or shoe brake is shown in Fig. 19.1. It consists of a block or shoe which is pressed against the rim of a revolving brake wheel drum. The block is made of a softer material than the rim of the wheel. This type of a brake is commonly used on railway trains and tram cars. The friction between the block and the wheel causes a tangential braking force to act on the wheel, which retard the rotation of the wheel. The block is pressed against the wheel by a force applied to one end of a lever to which the block is rigidly fixed as shown in Fig. 19.1. The other end of the lever is pivoted on a fixed fulcrum O.




(*a*) Clockwise rotation of brake wheel (*b*) Anticlockwise rotation of brake wheel.

**Fig. 19.1.** Single block brake. Line of action of tangential force passes through the fulcrum of the lever.

Let P = Force applied at the end of the lever,

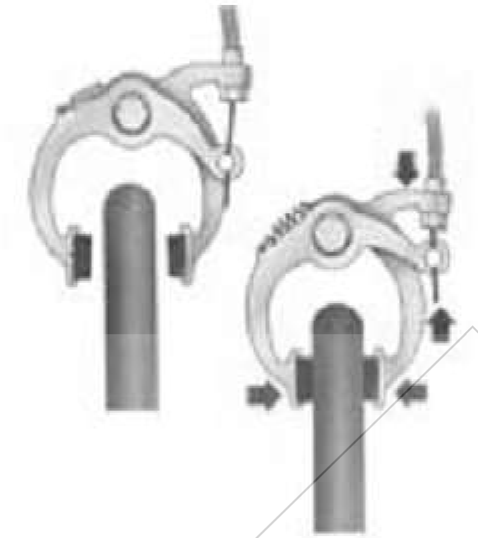
$R_N$= Normal force pressing the brake block on the wheel,

r = Radius of the wheel,

2θ = Angle of contact surface of the block,

μ = Coefficient of friction, and

$F_t$ = Tangential braking force or the frictional force acting at the contact surface of the block and the wheel.



When brakes are on, the pads grip the wheel rim from either side, friction between the pads and the rim converts the cycle's kinetic energy into heat as they reduce its speed.

If the angle of contact is less than 60°, then it may be assumed that the normal pressure between the block and the wheel is uniform. In such cases, tangential braking force on the wheel,

$$F_t = \mu.R_N \qquad ...(i)$$

and the braking torque,

$$T_B = F_t.r = \mu.R_N.r \qquad ... (ii)$$

Let us now consider the following three cases:

***Case 1***. When the line of action of tangential braking force ($F_t$) passes through the fulcrum O of the lever, and the brake wheel rotates clockwise as shown in Fig. 19.1 (a), then for equilibrium, taking moments about the fulcrum O, we have

$$R_N \times x = P \times l \quad \text{or} \quad R_N = \frac{P \times l}{x}$$

∴ Braking torque,

$$T_B = \mu.R_N.r = \mu \times \frac{P.l}{x} \times r = \frac{\mu.P.l.r}{x}$$

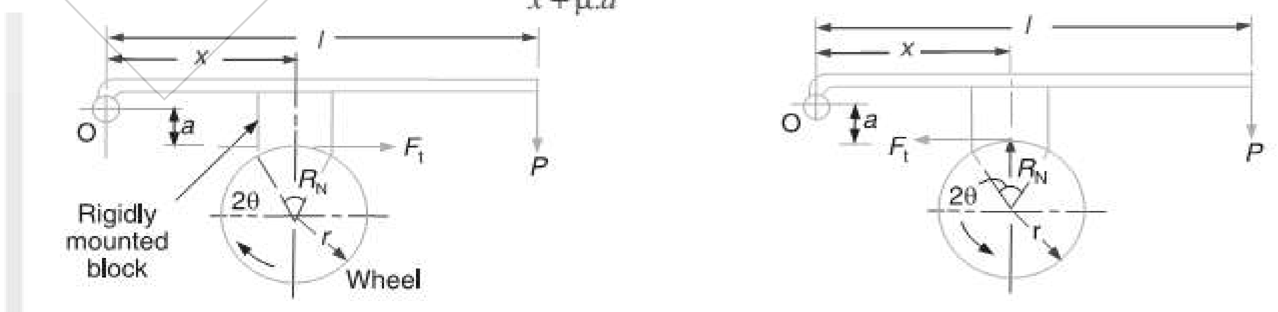
It may be noted that when the brake wheel rotates anticlockwise as shown in Fig. 9.1 (b), then the braking torque is same, i.e.

$$T_B = \mu.R_N.r = \frac{\mu.P.l.r}{x}$$

***Case 2***. When the line of action of the tangential braking force (Ft ) passes through a distance 'a' below the fulcrum O, and the brake wheel rotates clockwise as shown in Fig. 19.2 (a), then for equilibrium, taking moments about the fulcrum O,

$$R_N \times x + F_t \times a = P.l \quad \text{or} \quad R_N \times x + \mu R_N \times a = P.l \quad \text{or} \quad R_N = \frac{P.l}{x + \mu.a}$$

and braking torque, $T_B = \mu R_N.r = \dfrac{\mu.p.l.r}{x + \mu.a}$




(*a*) Clockwise rotation of brake wheel. (*b*) Anticlockwise rotation of brake wheel.

**Fig. 19.2.** Single block brake. Line of action of $F_t$ passes below the fulcrum.

When the brake wheel rotates anticlockwise, as shown in Fig. 19.2 ($b$), then for equilibrium,

$$R_N.x = P.l + F_t.a = P.l + \mu.R_N.a \qquad ...(i)$$

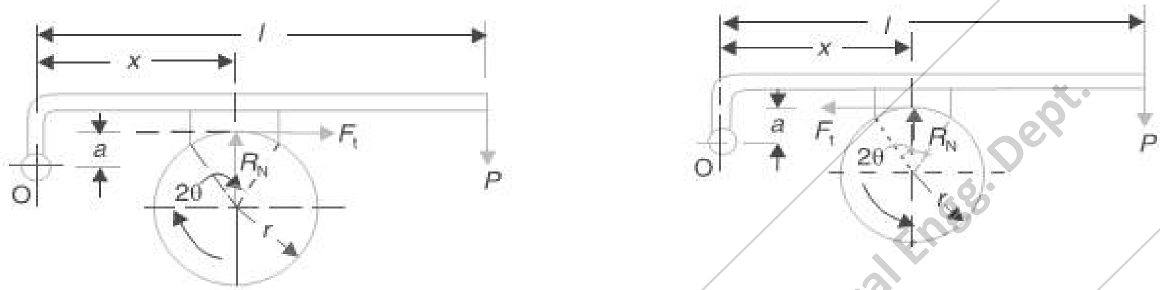
or $\quad R_N(x - \mu.a) = P.l \quad$ or $\quad R_N = \dfrac{P.l}{x - \mu.a}$

and braking torque, $\quad T_B = \mu.R_N.r = \dfrac{\mu.P.l.r}{x - \mu.a}$

**Case 3.** When the line of action of the tangential braking force ($F_t$) passes through a distance '$a$' above the fulcrum $O$, and the brake wheel rotates clockwise as shown in Fig. 19.3 ($a$), then for equilibrium, taking moments about the fulcrum $O$, we have

$$R_N.x = P.l + F_t\,.\,a = P.l + \mu.R_N.a \qquad ...(ii)$$

or $\quad R_N(x - \mu.a) = P.l \quad$ or $\quad R_N = \dfrac{P.l}{x - \mu.a}$




($a$) Clockwise rotation of brake wheel. ($b$) Anticlockwise rotation of brake wheel.

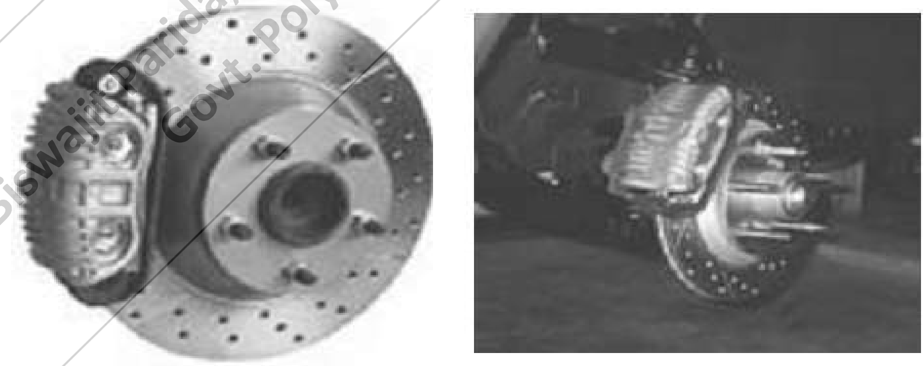
**Fig. 19.3.** Single block brake. Line of action of $F_t$ passes above the fulcrum.

and braking torque, $\quad T_B = \mu.R_N.r = \dfrac{\mu.P.l.r}{x - \mu.a}$

**Notes : 1.** From above we see that when the brake wheel rotates anticlockwise in case 2 [Fig. 19.2 ($b$)] and when it rotates clockwise in case 3 [Fig. 19.3 ($a$)], the equations ($i$) and ($ii$) are same, *i.e.*

$$R_N \times x = P.l + \mu.R_N.a$$

From this we see that the moment of frictional force ($\mu.R_N.a$) adds to the moment of force ($P.l$). In other words, the frictional force helps to apply the brake. Such type of brakes are said to be ***self energizing brakes***. When the frictional force is great enough to apply the brake with no external force, then the brake is said to be ***self-locking brake***.



Shoe brakes of a racing car

From the above expression, we see that if $x \le \mu.a$, then $P$ will be negative or equal to zero. This means no external force is needed to apply the brake and hence the brake is self locking. Therefore the condition for the brake to be self locking is

$$x \le \mu.a$$

The self locking brake is used only in back-stop applications.

**2.** The brake should be self energizing and not the self locking.

**3.** In order to avoid self locking and to prevent the brake from grabbing, $x$ is kept greater than $\mu\,.\,a$.

**4.** If $A_b$ is the projected bearing area of the block or shoe, then the bearing pressure on the shoe,

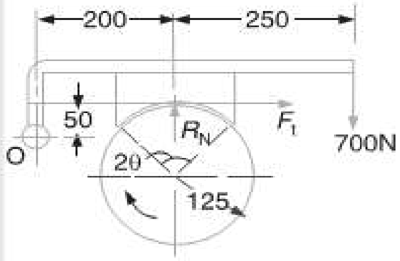
$$p_b = R_N / A_b$$

We know that $\quad A_b$ = Width of shoe × Projected length of shoe = $w(2r\sin\theta)$

**5.** When a single block or shoe brake is applied to a rolling wheel, an additional load is thrown on the shaft bearings due to heavy normal force ($R_N$) and produces bending of the shaft.

In order to overcome this drawback, a double block or shoe brake is used, as discussed in Art. 19.6.

**Example 19.1.** *A single block brake is shown in Fig. 19.5. The diameter of the drum is 250 mm and the angle of contact is 90°. If the operating force of 700 N is applied at the end of a lever and the coefficient of friction between the drum and the lining is 0.35, determine the torque that may be transmitted by the block brake.*




All dimensions in mm.
**Fig. 19.5**

**Solution.** Given : $d = 250$ mm or $r = 125$ mm ; $2\theta = 90° = \pi/2$ rad ; $P = 700$ N ; $\mu = 0.35$

Since the angle of contact is greater than 60°, therefore equivalent coefficient of friction,

$$\mu' = \frac{4\mu \sin\theta}{2\theta + \sin 2\theta} = \frac{4 \times 0.35 \times \sin 45°}{\pi/2 + \sin 90°} = 0.385$$

Let $R_N$ = Normal force pressing the block to the brake drum, and

$F_t$ = Tangential braking force = $\mu'.R_N$

Taking moments about the fulcrum $O$, we have

$$700(250+200) + F_t \times 50 = R_N \times 200 = \frac{F_t}{\mu'} \times 200 = \frac{F_t}{0.385} \times 200 = 520\, F_t$$

or $520\, F_t - 50 F_t = 700 \times 450$ or $F_t = 700 \times 450/470 = 670$ N

We know that torque transmitted by the block brake,

$$T_B = F_t \times r = 670 \times 125 = 8\,3750 \text{ N-mm} = 83.75\text{N-m}$$ **Ans.**

**Dynamometer:**

A dynamometer is a brake but in addition it has a device to measure the frictional resistance. Knowing the frictional resistance, we may obtain the torque transmitted and hence the power of the engine.

**Types of Dynamometer:**

1. ***Absorption dynamometer***, (In the absorption dynamometers, the entire energy or power produced by the engine are absorbed by the friction resistances of the brake and are transformed into heat, during the process of measurement.)
   - 1.1. Prony brake dynamometer
   - 1.2. Rope brake dynamometer
2. ***Transmission dynamometer***, (in the transmission dynamometers, the energy is not wasted in friction but is used for doing work. The energy or power produced by the engine is transmitted through the dynamometer to some other machines where the power developed is suitably measured.)
   - 2.1. Belt transmission dynamometer
   - 2.2. Epicyclic dynamometer
   - 2.3. Torsion dynamometer

**Prony Brake Dynamometer:**

It consists of two wooden blocks placed around a pulley fixed to the shaft of an engine whose power is required to be measured. The blocks are clamped by means of two bolts and nuts, as shown in Fig. 19.31. A helical spring is provided between the nut and the upper block to adjust the pressure on the pulley to control its speed. The upper block has a long lever attached to it and carries a weight W at its outer end. A counter weight is placed at the other end of the lever which balances the brake when unloaded. Two stops S, S are provided to limit the motion of the lever.

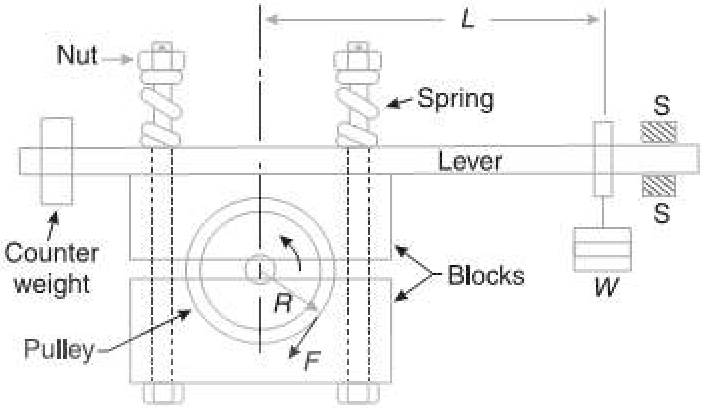



**Fig. 19.31.** Prony brake dynamometer.

When the brake is to be put in operation, the long end of the lever is loaded with suitable weights W and the nuts are tightened until the engine shaft runs at a constant speed and the lever is in horizontal position. Under these conditions, the moment due to the weight W must balance the moment of the frictional resistance between the blocks and the pulley.

Let $W$ = Weight at the outer end of the lever in newtons,

$L$ = Horizontal distance of the weight $W$ from the centre of the pulley in metres,

$F$ = Frictional resistance between the blocks and the pulley in newtons,

$R$ = Radius of the pulley in metres, and

$N$ = Speed of the shaft in r.p.m.

We know that the moment of the frictional resistance or torque on the shaft,

$T = W.L = F.R$ N-m

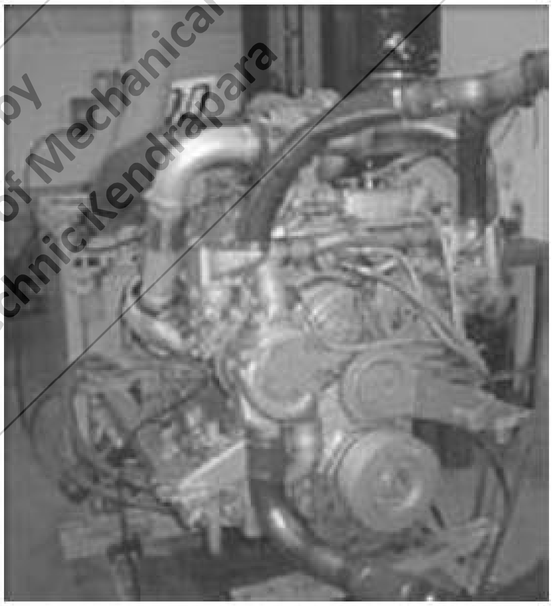
Work done in one revolution

= Torque × Angle turned in radians

$= T \times 2\pi$ N-m

∴ Work done per minute

$= T \times 2\pi N$ N-m



Another dynamo

We know that brake power of the engine,

$$B.P. = \frac{\text{Work done per min.}}{60} = \frac{T \times 2\pi N}{60} = \frac{W.L \times 2\pi N}{60} \text{ watts}$$

**Rope Brake Dynamometer:**

It is another form of absorption type dynamometer which is most commonly used for measuring the brake power of the engine. It consists of one, two or more ropes wound around the flywheel or rim of a pulley fixed rigidly to the shaft of an engine. The upper end of the ropes is attached to a spring balance while the lower end of the ropes is kept in position by applying a dead weight as shown in Fig. 19.32. In order to prevent the slipping of the rope over the flywheel, wooden blocks are placed at intervals around the circumference of the flywheel.

In the operation of the brake, the engine is made to run at a constant speed. The frictional torque, due to the rope, must be equal to the torque being transmitted by the engine.

Let W = Dead load in newtons,
S = Spring balance reading in newtons,
D = Diameter of the wheel in metres,
d = diameter of rope in metres, and
N = Speed of the engine shaft in r.p.m.

∴ Net load on the brake

$$= (W - S)\ N$$

We know that distance moved in one revolution

$$= \pi\ (D + d)\ m$$

∴ Work done per revolution

$$= (W - S)\ \pi\ (D + d)\ \text{N-m}$$

and work done per minute

$$= (W - S)\ \pi\ (D + d)\ N\ \text{N-m}$$

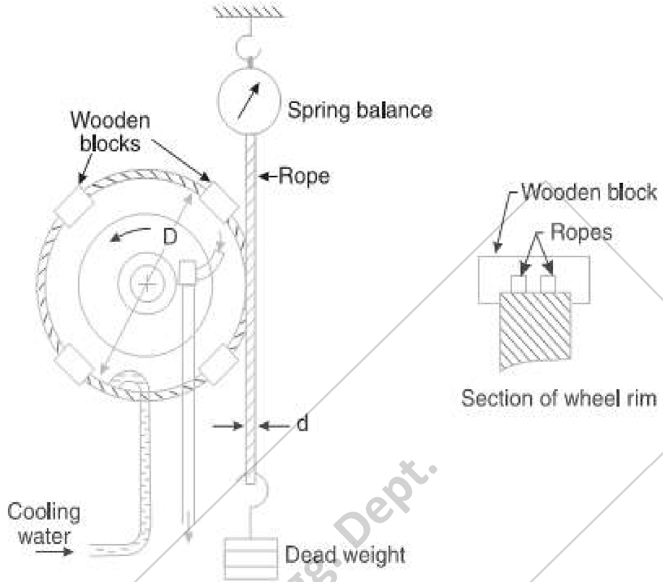


Fig. 19.32. Rope brake dynamometer.

∴ Brake power of the engine,

$$B.P = \frac{\text{Work done per min}}{60} = \frac{(W - S)\ \pi\ (D + d) N}{60}\ \text{watts}$$

If the diameter of the rope (d) is neglected, then brake power of the engine,

$$B.P. = \frac{(W - S)\ \pi D\ N}{60}\ \text{watts}$$

**Note:** Since the energy produced by the engine is absorbed by the frictional resistances of the brake and is transformed into heat, therefore it is necessary to keep the flywheel of the engine cool with soapy water. The flywheels have their rims made of a channel section so as to receive a stream of water which is being whirled round by the wheel. The water is kept continually flowing into the rim and is drained away by a sharp edged scoop on the other side, as shown in Fig. 19.32.

**Example 19.17.** *In a laboratory experiment, the following data were recorded with rope brake: Diameter of the flywheel 1.2 m; diameter of the rope 12.5 mm; speed of the engine 200 r.p.m.; dead load on the brake 600 N; spring balance reading 150 N. Calculate the brake power of the engine.*

**Solution.** Given: D = 1.2 m; d = 12.5 mm = 0.0125 m; N = 200 r.p.m; W = 600 N; S = 150 N

We know that brake power of the engine,

$$B.P. = \frac{(W - S)\ \pi\ (D + d) N}{60} = \frac{(600 - 150)\ \pi\ (1.2 + 0.0125)200}{60} = 5715\ W$$

$$= 5.715\ \text{kW}\ \textbf{Ans.}$$

# Chapter-03: Power Transmission

**Power:** It is the rate of doing work. Power is defined formally as units of energy per unit time.

In SI units: Watt = Joule/Second = (Newton x Meter) / Second

**Power Transmission:**

- Power transmission is the movement of energy from its place of generation to a location where it is applied to perform useful work.
- Power transmission is a process to transmit motion from one shaft to another by using some connection between them like belt, rope, chain, and gears.
- To connect the shafts, mainly two types of connectors are used, one is flexible and other is rigid. In flexible types of connection, there is relative velocity between shaft and connectors due to slip and strain produced in the connectors. But in case of rigid connection, there is no relative velocity between the connector and shaft.
- When the power is transmitted from input to output using mechanical elements is known as *mechanical power transmission*.

**Types of Mechanical Drives:**

A. ***Belt Drive*** (To transmit power from one shaft to another, pulleys are mounted on the two shafts. The pulleys are then connected by an endless belt passing over the pulleys. The connecting belt is kept in tension so that motion of one pulley is transferred to the other without slip.)



**Advantages**

- Made from flexible material and can be used for the shafts whose axes are not parallel.
- Initial cost is low.
- Lubricant is not required.
- Noise is relatively less.
- It is a flexible member; it slips or breaks during overload and safeguards the machine.

**Disadvantages**

- Drive is not positive as the belt slip over the pulleys.
- Occupies relatively more space.
- Speed ratio cannot be maintained because of slipping of belt.
- Adjust the tension in the belt is require time to time.
- Life is relatively low.

B. ***Gear Drive*** (In gear drive no. of teeth are cut on both the blanks of the gear wheel which mesh with each other. The projections on one disc and recesses on other disc are made to mesh with each other to avoid slipping.)

**Advantages**

- Positive drive and has more efficiency than belt and rope drive.
- Operation of drive is simple and effective.
- Life is more compared to other drives.

- With one input speed, no. of output speeds can be obtained by using suitable gear drive.
- Safe and compact.
- Constant velocity ration is obtained.
- Using different type of gears, power can be transmitted between shafts whose axes are parallel, inclined or intersecting to each other.

**Disadvantages**

- If the tooth geometry of the gear is not properly maintained, the drive may get locked.
- Not preferred when very high speed transmission is required.
- If lubrication arrangement is not provided, it may produce noise.

***C. Chain Drive*** (The chain drive consist of three elements – driving sprocket, driven sprocket and endless chain wrapped around the sprocket. The chain drive is positive drive where there is no slip & constant velocity ratio can be maintained)

**Advantages**

- Positive drive as there is no slip, hence constant velocity ratio.
- Occupies less space compared to belt drive.
- Life is more compared to belt drive.
- Used for large center distance.
- Transmission efficiency is larger than belt drive.

**Disadvantages**

- Noisy compared to belt drive.
- Initial cost is higher compared to belt drive
- Adjustment in the center distance is necessary.
- Maintenance cost is higher & complex compared to belt drive.

**Types of Belt Drives:**

1. ***Light drive:*** These are used to transmit small powers at belt speeds up to about 10 m/s, as in agricultural machines and small machine tools.
2. ***Medium drives:*** These are used to transmit medium power at belt speeds over 10 m/s but up to 22 m/s, as in machine tools.
3. ***Heavy drives:*** These are used to transmit large powers at belt speeds above 22 m/s, as in compressors and generators.

**Types of Belts:**

a) ***Flat Belt:*** The flat belt is mostly used in the factories and workshops, where a moderate amount of power is to be transmitted, from one pulley to another when the two pulleys are not more than 8 meters apart.

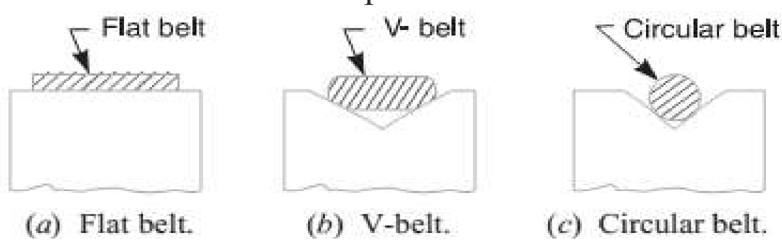



**Fig. 11.1.** Types of belts.

*b) V-belt:* The V-belt is mostly used in the factories and workshops, where a moderate amount of power is to be transmitted, from one pulley to another, when the two pulleys are very near to each other.

*c) Circular belt or rope:* The circular belt or rope is mostly used in the factories and workshops, where a great amount of power is to be transmitted, from one pulley to another, when the two pulleys are more than 8 meters apart.

**Types of Flat Belt Drives:**

***1. Open Belt drive:*** The open belt drive is used with shafts arranged parallel and rotating in the same direction. In this case, the driver A pulls the belt from one side (i.e. lower side RQ) and delivers it to the other side (i.e. upper side LM). Thus the tension in the lower side belt will be more than that in the upper side belt. The lower side belt (because of more tension) is known as *tight side* whereas the upper side belt (because of less tension) is known as *slack side*.

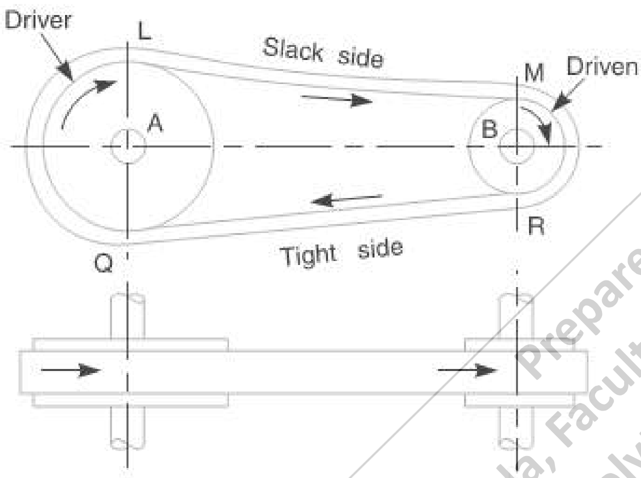



Fig. 11.3. Open belt drive.

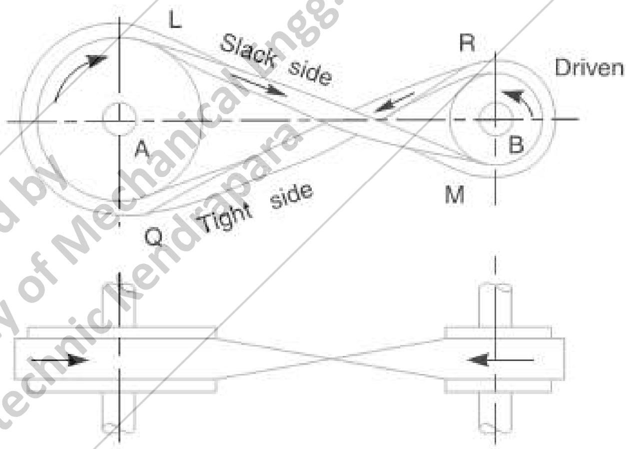



Fig. 11.4. Crossed or twist belt drive.



***2. Crossed or twist Belt drive:*** The crossed or twist belt drive is used with shafts arranged parallel and rotating in the opposite directions. In this case, the driver pulls the belt from one side (i.e. RQ) and delivers it to the other side (i.e. LM). Thus the tension in the belt RQ will be more than that in the belt LM. The belt RQ (because of more tension) is known as *tight side*, whereas the belt LM (because of less tension) is known as *slack side*. A little consideration will show that at a point where the belt crosses, it rubs against each other and there will be excessive wear and tear. In order to avoid this, the shafts should be placed at a maximum distance of 20 b, where b is the width of belt and the speed of the belt should be less than 15 m/s.

**Velocity Ratio of Belt Drive:**

It is the ratio between the velocities of the driver and the follower or driven.

Let $d_1$ = Diameter of the driver, $d_2$= Diameter of the follower,

$N_1$= Speed of the driver in r.p.m., and $N_2$= Speed of the follower in r.p.m.

$\therefore$ Length of the belt that passes over the driver, in one minute

$$= \pi\, d_1.N_1$$

Similarly, length of the belt that passes over the follower, in one minute

$$= \pi\, d_2.N_2$$

Since the length of belt that passes over the driver in one minute is equal to the length of belt that passes over the follower in one minute, therefore

$$\pi\, d_1.N_1 = \pi\, d_2.N_2$$

$\therefore$ Velocity ratio, $\frac{N_2}{N_1} = \frac{d_1}{d_2}$

When the thickness of the belt ($t$) is considered, then velocity ratio,

$$\frac{N_2}{N_1} = \frac{d_1 + t}{d_2 + t}$$

**Velocity Ratio of a Compound Belt Drive:**

Sometimes the power is transmitted from one shaft to another, through a number of pulleys. Consider a pulley 1 driving the pulley 2. Since the pulleys 2 and 3 are keyed to the same shaft, therefore the pulley 1 also drives the pulley 3 which, in turn, drives the pulley 4.

Let $d_1$ = Diameter of the pulley 1, $N_1$= Speed of the pulley 1 in r.p.m.,

$d_2$, $d_3$, $d_4$, and $N_2$, $N_3$, $N_4$ = Corresponding values for pulleys 2, 3 and 4.

We know that velocity ratio of pulleys 1 and 2,

$$\frac{N_2}{N_1} = \frac{d_1}{d_2} \quad ...(i)$$

Similarly, velocity ratio of pulleys 3 and 4,

$$\frac{N_4}{N_3} = \frac{d_3}{d_4} \quad ...(ii)$$

Multiplying equations (*i*) and (*ii*),

$$\frac{N_2}{N_1} \times \frac{N_4}{N_3} = \frac{d_1}{d_2} \times \frac{d_3}{d_4}$$

or

$$\frac{N_4}{N_1} = \frac{d_1 \times d_3}{d_2 \times d_4} \quad ...(\because N_2 = N_3, \text{ being keyed to the same shaft})$$

A little consideration will show, that if there are six pulleys, then

$$\frac{N_6}{N_1} = \frac{d_1 \times d_3 \times d_5}{d_2 \times d_4 \times d_6}$$

or

$$\frac{\text{Speed of last driven}}{\text{Speed of first driver}} = \frac{\text{Product of diameters of drivers}}{\text{Product of diameters of drivens}}$$

**Slip of Belt:**

Sometimes the frictional grip between the belts and the shafts becomes insufficient, it may cause some forward motion of the driver without carrying the belt with it. This may also cause some forward motion of the belt without carrying the driven pulley with it. This is called ***slip*** of the belt and is generally expressed as a percentage.

The result of the belt slipping is to reduce the velocity ratio of the system. Thus the belt should never be used where a definite velocity ratio is of importance (as in the case of hour, minute and second arms in a watch).

Let $s_1$ % = Slip between the driver and the belt, and

$s_2$ % = Slip between the belt and the follower.

∴ Velocity of the belt passing over the driver per second

$$v = \frac{\pi d_1 . N_1}{60} - \frac{\pi d_1 . N_1}{60} \times \frac{s_1}{100} = \frac{\pi d_1 . N_1}{60}\left(1 - \frac{s_1}{100}\right) \qquad ...(i)$$

and velocity of the belt passing over the follower per second,

$$\frac{\pi d_2 . N_2}{60} = v - v \times \frac{s_2}{100} = v\left(1 - \frac{s_2}{100}\right)$$

Substituting the value of $v$ from equation $(i)$,

$$\frac{\pi d_2 \, N_2}{60} = \frac{\pi d_1 \, N_1}{60}\left(1 - \frac{s_1}{100}\right)\left(1 - \frac{s_2}{100}\right)$$

$$\frac{N_2}{N_1} = \frac{d_1}{d_2}\left(1 - \frac{s_1}{100} - \frac{s_2}{100}\right) \qquad ...\left(\text{Neglecting } \frac{s_1 \times s_2}{100 \times 100}\right)$$

$$= \frac{d_1}{d_2}\left(1 - \frac{s_1 + s_2}{100}\right) = \frac{d_1}{d_2}\left(1 - \frac{s}{100}\right)$$

...(where $s = s_1 + s_2$, *i.e.* total percentage of slip)

If thickness of the belt ($t$) is considered, then

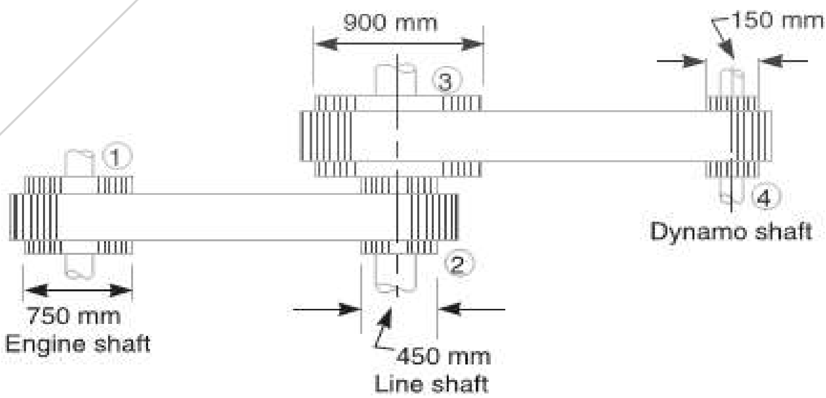
$$\frac{N_2}{N_1} = \frac{d_1 + t}{d_2 + t}\left(1 - \frac{s}{100}\right)$$

**Example 11.1.** *An engine, running at 150 r.p.m., drives a line shaft by means of a belt. The engine pulley is 750 mm diameter and the pulley on the line shaft being 450 mm. A 900 mm diameter pulley on the line shaft drives a 150 mm diameter pulley keyed to a dynamo shaft. Find the speed of the dynamo shaft, when* **1.** *there is no slip, and* **2.** *there is a slip of 2% at each drive.*

**Solution.** Given : $N_1$ = 150 r.p.m. ; $d_1$ = 750 mm ; $d_2$ = 450 mm ; $d_3$ = 900 mm ; $d_4$ = 150 mm

The arrangement of belt drive is shown in Fig. 11.10.

Let $N_4$ = Speed of the dynamo shaft .



**1. *When there is no slip***

We know that $\frac{N_4}{N_1} = \frac{d_1 \times d_3}{d_2 \times d_4}$ or $\frac{N_4}{150} = \frac{750 \times 900}{450 \times 150} = 10$

$\therefore$ $N_4 = 150 \times 10 = 1500$ r.p.m. **Ans.**

**2. *When there is a slip of 2% at each drive***

We know that $\frac{N_4}{N_1} = \frac{d_1 \times d_3}{d_2 \times d_4}\left(1 - \frac{s_1}{100}\right)\left(1 - \frac{s_2}{100}\right)$

$$\frac{N_4}{150} = \frac{750 \times 900}{450 \times 150}\left(1 - \frac{2}{100}\right)\left(1 - \frac{2}{100}\right) = 9.6$$

$\therefore$ $N_4 = 150 \times 9.6 = 1440$ r.p.m. **Ans.**

## Length of an Open Belt Drive:

Let $r_1$ and $r_2$ = Radii of the larger and smaller pulleys,

$x$ = Distance between the centres of two pulleys (i.e. $O_1O_2$), and

$L$ = Total length of the belt.

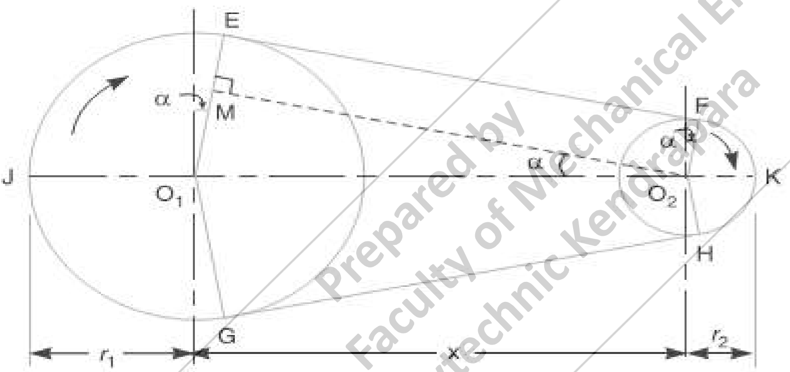



Fig. 11.11. Length of an open belt drive.



Let the belt leaves the larger pulley at E and G and the smaller pulley at F and H. Through $O_2$, draw $O_2M$ parallel to FE.

From the geometry of the figure, we find that $O_2M$ will be perpendicular to $O_1E$.

Let the angle $MO_2O_1 = \alpha$ radians.

We know that the length of the belt,

$$L = Arc\ GJE + EF + Arc\ FKH + HG$$

$$= 2\,(Arc\ JE + EF + Arc\ FK) \quad \ldots(i)$$

From the geometry of the figure, we find that

$$\sin\alpha = \frac{O_1M}{O_1O_2} = \frac{O_1E - EM}{O_1O_2} = \frac{r_1 - r_2}{x}$$

Since $\alpha$ is very small, therefore putting

$$\sin\alpha = \alpha \text{ (in radians)} = \frac{r_1 - r_2}{x} \quad \ldots(ii)$$

$$\therefore \quad Arc\ JE = r_1\left(\frac{\pi}{2} + \alpha\right) \quad \ldots(iii)$$

Similarly $\quad \text{Arc } FK = r_2\left(\frac{\pi}{2} - \alpha\right) \quad \ldots(iv)$

and

$$EF = MO_2 = \sqrt{(O_1O_2)^2 - (O_1M)^2} = \sqrt{x^2 - (r_1 - r_2)^2}$$

$$= x\sqrt{1 - \left(\frac{r_1 - r_2}{x}\right)^2}$$

Expanding this equation by binomial theorem,

$$EF = x\left[1 - \frac{1}{2}\left(\frac{r_1 - r_2}{x}\right)^2 + \ldots\right] = x - \frac{(r_1 - r_2)^2}{2x} \quad \ldots(v)$$

Substituting the values of arc *JE* from equation (*iii*), arc *FK* from equation (*iv*) and *EF* from equation (*v*) in equation (*i*), we get

$$L = 2\left[r_1\left(\frac{\pi}{2} + \alpha\right) + x - \frac{(r_1 - r_2)^2}{2x} + r_2\left(\frac{\pi}{2} - \alpha\right)\right]$$

$$= 2\left[r_1 \times \frac{\pi}{2} + r_1.\alpha + x - \frac{(r_1 - r_2)^2}{2x} + r_2 \times \frac{\pi}{2} - r_2.\alpha\right]$$

$$= 2\left[\frac{\pi}{2}(r_1 + r_2) + \alpha(r_1 - r_2) + x - \frac{(r_1 - r_2)^2}{2x}\right]$$

$$= \pi(r_1 + r_2) + 2\alpha(r_1 - r_2) + 2x - \frac{(r_1 - r_2)^2}{x}$$

Substituting the value of $\alpha = \frac{r_1 - r_2}{x}$ from equation (*ii*),

$$L = \pi(r_1 + r_2) + 2 \times \frac{(r_1 - r_2)}{x} \times (r_1 - r_2) + 2x - \frac{(r_1 - r_2)^2}{x}$$

$$= \pi(r_1 + r_2) + \frac{2(r_1 - r_2)^2}{x} + 2x - \frac{(r_1 - r_2)^2}{x}$$

$$= \pi(r_1 + r_2) + 2x + \frac{(r_1 - r_2)^2}{x} \quad \ldots\text{(In terms of pulley radii)}$$

$$= \frac{\pi}{2}(d_1 + d_2) + 2x + \frac{(d_1 - d_2)^2}{4x} \quad \ldots\text{(In terms of pulley diameters)}$$

## Length of a Cross Belt Drive:

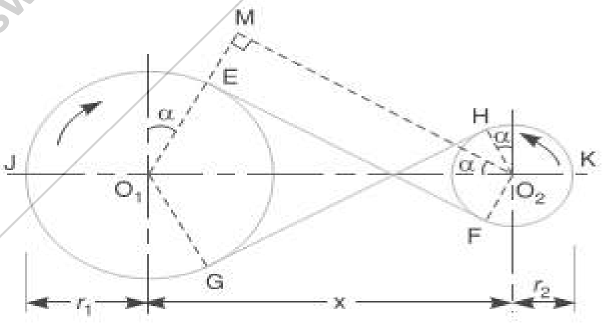



**Fig. 11.12.** Length of a cross belt drive.

Let the belt leaves the larger pulley at E and G and the smaller pulley at F and H. Through $O_2$, draw $O_2M$ parallel to FE.

From the geometry of the figure, we find that $O_2M$ will be perpendicular to $O_1E$.

Let the angle $MO_2O_1 = \alpha$ radians.

We know that the length of the belt,

$$L = Arc\ GJE + EF + Arc\ FKH + HG$$
$$= 2\ (Arc\ JE + EF + Arc\ FK) \quad ...(i)$$

From the geometry of the figure, we find that

$$\sin\alpha = \frac{O_1 M}{O_1 O_2} = \frac{O_1 E + EM}{O_1 O_2} = \frac{r_1 + r_2}{x}$$

Since α is very small, therefore putting

$$\sin\alpha = \alpha \text{ (in radians)} = \frac{r_1 + r_2}{x} \quad ...(ii)$$

$$\therefore \quad \text{Arc } JE = r_1\left(\frac{\pi}{2} + \alpha\right) \quad ...(iii)$$

Similarly $\quad \text{Arc } FK = r_2\left(\frac{\pi}{2} + \alpha\right) \quad ...(iv)$

and $\quad EF = MO_2 = \sqrt{(O_1O_2)^2 - (O_1 M)^2} = \sqrt{x^2 - (r_1 + r_2)^2}$

$$= x\sqrt{1 - \left(\frac{r_1 + r_2}{x}\right)^2}$$

Expanding this equation by binomial theorem,

$$EF = x\left[1 - \frac{1}{2}\left(\frac{r_1 + r_2}{x}\right)^2 + ...\right] = x - \frac{(r_1 + r_2)^2}{2x} \quad ...(v)$$

Substituting the values of arc $JE$ from equation (*iii*), arc $FK$ from equation (*iv*) and $EF$ from equation (*v*) in equation (*i*), we get

$$L = 2\left[r_1\left(\frac{\pi}{2} + \alpha\right) + x - \frac{(r_1 + r_2)^2}{2x} + r_2\left(\frac{\pi}{2} + \alpha\right)\right]$$

$$= 2\left[r_1 \times \frac{\pi}{2} + r_1.\alpha + x - \frac{(r_1 + r_2)^2}{2x} + r_2 \times \frac{\pi}{2} + r_2.\alpha\right]$$

$$= 2\left[\frac{\pi}{2}(r_1 + r_2) + \alpha(r_1 + r_2) + x - \frac{(r_1 + r_2)^2}{2x}\right]$$

$$= \pi(r_1 + r_2) + 2\alpha(r_1 + r_2) + 2x - \frac{(r_1 + r_2)^2}{x}$$

Substituting the value of $\alpha = \frac{r_1 + r_2}{x}$ from equation (*ii*),

$$L = \pi(r_1 + r_2) + \frac{2(r_1 + r_2)}{x} \times (r_1 + r_2) + 2x - \frac{(r_1 + r_2)^2}{x}$$

$$= \pi(r_1 + r_2) + \frac{2(r_1 + r_2)^2}{x} + 2x - \frac{(r_1 + r_2)^2}{x}$$

$$= \pi(r_1 + r_2) + 2x + \frac{(r_1 + r_2)^2}{x} \quad \text{...(In terms of pulley radii)}$$

$$= \frac{\pi}{2}(d_1 + d_2) + 2x + \frac{(d_1 + d_2)^2}{4x} \quad \text{...(In terms of pulley diameters)}$$

**Power Transmitted by a Belt:**

Let $T_1$ and $T_2$ = Tensions in the tight and slack side of the belt respectively in newtons,
$r_1$ and $r_2$ = Radii of the driver and follower respectively, and
v = Velocity of the belt in m/s.

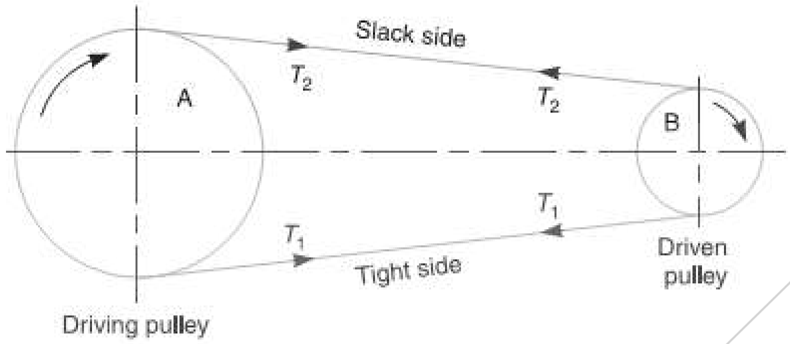



Fig. 11.14. Power transmitted by a belt.

The effective turning (driving) force at the circumference of the follower is the difference between the two tensions (i.e. $T_1 – T_2$).

∴ Work done per second = $(T_1 – T_2)$ v N-m/s

and Power transmitted, P = $(T_1 – T_2)$ v W ...(∵ 1 N-m/s = 1 W)

A little consideration will show that the torque exerted on the driving pulley is $(T_1–T_2)$ $r_1$. Similarly, the torque exerted on the driven pulley i.e. follower is $(T_1 – T_2)$ $r_2$.

**Ratio of Driving Tensions for Flat Belt Drive:**

Consider a driven pulley rotating in the clockwise direction

Let $T_1$ and $T_2$ = Tensions in the tight and slack side of the belt respectively,
θ = Angle of contact in radians (i.e. angle subtended by the arc AB, along which the belt touches the pulley at the centre).

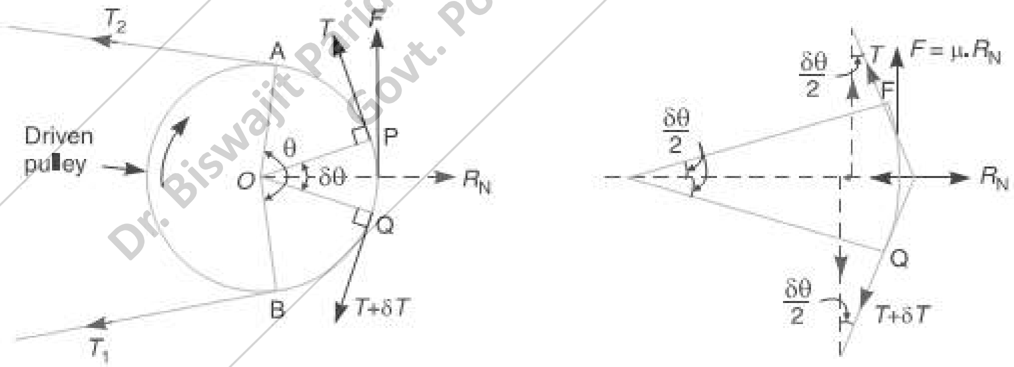



Fig. 11.15. Ratio of driving tensions for flat belt.

Now consider a small portion of the belt PQ, subtending an angle δθ at the centre of the pulley. The belt PQ is in equilibrium under the following forces :

1. Tension T in the belt at P,
2. Tension (T + δT) in the belt at Q,
3. Normal reaction $R_N$, and
4. Frictional force, $F = \mu \times R_N$, where μ is the coefficient of friction between the belt and pulley

Resolving all the forces horizontally and equating the same,

$$R_N = (T + \delta T) \sin \frac{\delta\theta}{2} + T \sin \frac{\delta\theta}{2} \qquad ...(i)$$

Since the angle δθ is very small, therefore putting $\sin \delta\theta / 2 = \delta\theta / 2$ in equation (*i*),

$$R_N = (T + \delta T) \frac{\delta\theta}{2} + T \times \frac{\delta\theta}{2} = \frac{T.\delta\theta}{2} + \frac{\delta T.\delta\theta}{2} + \frac{T.\delta\theta}{2} = T.\delta\theta \qquad ...(ii)$$

$$...\left(\text{Neglecting } \frac{\delta T.\delta\theta}{2}\right)$$

Now resolving the forces vertically, we have

$$\mu \times R_N = (T + \delta T) \cos \frac{\delta\theta}{2} - T \cos \frac{\delta\theta}{2} \qquad ...(iii)$$

Since the angle δθ is very small, therefore putting $\cos \delta\theta / 2 = 1$ in equation (*iii*),

$$\mu \times R_N = T + \delta T - T = \delta T \quad \text{or} \quad R_N = \frac{\delta T}{\mu} \qquad ...(iv)$$

Equating the values of $R_N$ from equations (*ii*) and (*iv*),

$$T.\delta\theta = \frac{\delta T}{\mu} \quad \text{or} \quad \frac{\delta T}{T} = \mu.\delta\theta$$

Integrating both sides between the limits $T_2$ and $T_1$ and from 0 to θ respectively,

*i.e.*
$$\int_{T_2}^{T_1} \frac{\delta T}{T} = \mu \int_0^{\theta} \delta\theta \quad \text{or} \quad \log_e \left(\frac{T_1}{T_2}\right) = \mu.\theta \quad \text{or} \quad \frac{T_1}{T_2} = e^{\mu.\theta} \qquad ...(v)$$

Equation (*v*) can be expressed in terms of corresponding logarithm to the base 10, *i.e.*

$$2.3 \log \left(\frac{T_1}{T_2}\right) = \mu.\theta$$

The above expression gives the relation between the tight side and slack side tensions, in terms of coefficient of friction and the angle of contact.

**Example 11.4.** *Find the power transmitted by a belt running over a pulley of 600 mm diameter at 200 r.p.m. The coefficient of friction between the belt and the pulley is 0.25, angle of lap 160° and maximum tension in the belt is 2500 N.*

**Solution.** Given : $d = 600$ mm $= 0.6$ m ; $N = 200$ r.p.m. ; $\mu = 0.25$ ; $\theta = 160° = 160 \times \pi / 180 = 2.793$ rad ; $T_1 = 2500$ N

We know that velocity of the belt,

$$v = \frac{\pi d.N}{60} = \frac{\pi \times 0.6 \times 200}{60} = 6.284 \text{ m/s}$$

Let $T_2$ = Tension in the slack side of the belt.

We know that
$$2.3 \log \left(\frac{T_1}{T_2}\right) = \mu.\theta = 0.25 \times 2.793 = 0.6982$$

$$\log \left(\frac{T_1}{T_2}\right) = \frac{0.6982}{2.3} = 0.3036$$

∴
$$\frac{T_1}{T_2} = 2.01 \qquad ...(\text{Taking antilog of } 0.3036)$$

and
$$T_2 = \frac{T_1}{2.01} = \frac{2500}{2.01} = 1244 \text{ N}$$

We know that power transmitted by the belt,

$$P = (T_1 - T_2)\, v = (2500 - 1244)\, 6.284 = 7890 \text{ W}$$
$$= 7.89 \text{ kW} \quad \textbf{Ans.}$$

**Example 11.5.** *A casting weighing 9 kN hangs freely from a rope which makes 2.5 turns round a drum of 300 mm diameter revolving at 20 r.p.m. The other end of the rope is pulled by a man. The coefficient of friction is 0.25. Determine* **1.** *The force required by the man, and* **2.** *The power to raise the casting.*

**Solution.** Given : $W = T_1 = 9$ kN = 9000 N ; $d = 300$ mm = 0.3 m ; $N = 20$ r.p.m. ; $\mu = 0.25$

**1.** ***Force required by the man***

Let $T_2$ = Force required by the man.

Since the rope makes 2.5 turns round the drum, therefore angle of contact,

$$\theta = 2.5 \times 2\pi = 5\pi \text{ rad}$$

We know that
$$2.3 \log\left(\frac{T_1}{T_2}\right) = \mu.\theta = 0.25 \times 5\pi = 3.9275$$

$$\log\left(\frac{T_1}{T_2}\right) = \frac{3.9275}{2.3} = 1.71 \quad \text{or} \quad \frac{T_1}{T_2} = 51$$

...(Taking antilog of 1.71)

$$\therefore \quad T_2 = \frac{T_1}{51} = \frac{9000}{51} = 176.47 \text{ N} \quad \textbf{Ans.}$$

**2.** ***Power to raise the casting***

We know that velocity of the rope,

$$v = \frac{\pi d.N}{60} = \frac{\pi \times 0.3 \times 20}{60} = 0.3142 \text{ m/s}$$

$\therefore$ Power to raise the casting,

$$P = (T_1 - T_2)\, v = (9000 - 176.47)\, 0.3142 = 2772 \text{ W}$$
$$= 2.772 \text{ kW} \quad \textbf{Ans.}$$



**Centrifugal Tension:**

Since the belt continuously runs over the pulleys, therefore, some centrifugal force is caused, whose effect is to increase the tension on both, tight as well as the slack sides. The tension caused by centrifugal force is called centrifugal tension.

Consider a small portion PQ of the belt subtending an angle dθ the centre of the pulley.

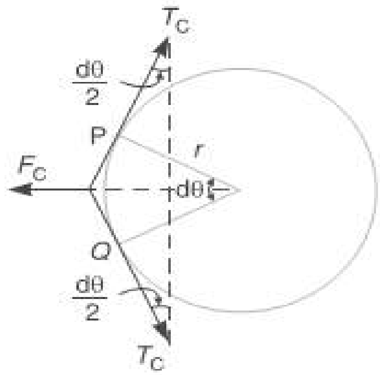



Fig. 11.17. Centrifugal tension.

Let m = Mass of the belt per unit length in kg,

v = Linear velocity of the belt in m/s,

r = Radius of the pulley over which the belt runs in metres, and

$T_C$= Centrifugal tension acting tangentially at P and Q in newtons.

We know that length of the belt PQ = r. dθ and mass of the belt PQ = m. r. dθ

∴ Centrifugal force acting on the belt PQ,

$$F_C = (m.r.d\theta)\frac{v^2}{r} = m.d\theta.v^2$$

The centrifugal tension $T_C$ acting tangentially at P and Q keeps the belt in equilibrium.

Now resolving the forces (i.e. centrifugal force and centrifugal tension) horizontally and equating the same, we have

$$T_C \sin\left(\frac{d\theta}{2}\right) + T_C \sin\left(\frac{d\theta}{2}\right) = F_C = m.d\theta.v^2$$

Since the angle $d\theta$ is very small, therefore, putting $\sin\left(\frac{d\theta}{2}\right) = \frac{d\theta}{2}$, in the above expression,

$$2T_C\left(\frac{d\theta}{2}\right) = m.d\theta.v^2 \quad \text{or} \quad T_C = m \,.\, v^2$$

**Notes : 1.** When the centrifugal tension is taken into account, then total tension in the tight side,

$$T_{t1} = T_1 + T_C$$

and total tension in the slack side,

$$T_{t2} = T_2 + T_C$$

**2.** Power transmitted,

$$P = (T_{t1} - T_{t2})v \quad \text{...(in watts)}$$

$$= [(T_1 + T_C) - (T_2 + T_C)]\, v = (T_1 - T_2)\, v \quad \text{...(same as before)}$$

Thus we see that centrifugal tension has no effect on the power transmitted.

**3.** The ratio of driving tensions may also be written as

$$2.3\log\left(\frac{T_{t1} - T_C}{T_{t2} - T_C}\right) = \mu.\theta$$

where $T_{t1}$ = Maximum or total tension in the belt.

**Maximum Tension in the Belt:**

A little consideration will show that the maximum tension in the belt (T) is equal to the total tension in the tight side of the belt ($T_{t1}$).

Let σ = Maximum safe stress in N/mm$^2$,

b = Width of the belt in mm, and

t = Thickness of the belt in mm.

We know that maximum tension in the belt,

T = Maximum stress × cross-sectional area of belt = σ. b. t

When centrifugal tension is neglected, then

T (or $T_{t1}$) = $T_1$, i.e. Tension in the tight side of the belt

and when centrifugal tension is considered, then

T (or $T_{t1}$) = $T_1 + T_C$

**Condition for the Transmission of Maximum Power:**

We know that power transmitted by a belt,

$$P = (T_1 - T_2)\, v \qquad ...(i)$$

where $T_1$ = Tension in the tight side of the belt in newtons,

$T_2$ = Tension in the slack side of the belt in newtons, and

$v$ = Velocity of the belt in m/s

From Art. 11.14, we have also seen that the ratio of driving tensions is

$$\frac{T_1}{T_2} = e^{\mu.\theta} \quad \text{or} \quad T_2 = \frac{T_1}{e^{\mu.\theta}} \qquad ...(ii)$$

Substituting the value of $T_2$ in equation (*i*),

$$P = \left(T_1 - \frac{T_1}{e^{\mu.\theta}}\right) v = T_1\left(1 - \frac{1}{e^{\mu.\theta}}\right) v = T_1 . v . C \qquad ...(iii)$$

where $$C = 1 - \frac{1}{e^{\mu.\theta}}$$

We know that $T_1 = T - T_C$

where $T$ = Maximum tension to which the belt can be subjected in newtons, and

$T_C$ = Centrifugal tension in newtons.

Substituting the value of $T_1$ in equation (*iii*),

$$P = (T - T_C)\, v.C$$

$$= (T - m.v^2)\, v.C = (T.v - m\, v^3)\, C \qquad \text{... (Substituting } T_C = m.\, v^2)$$

For maximum power, differentiate the above expression with respect to $v$ and equate to zero, *i.e.*

$$\frac{dP}{dv} = 0 \quad \text{or} \quad \frac{d}{dv}(T.v - mv^3)\, C = 0$$

$$\therefore \quad T - 3\, m\, .\, v^2 = 0$$

or $$T - 3\, T_C = 0 \quad \text{or} \quad T = 3\, T_C \qquad ...(iv)$$

It shows that when the power transmitted is maximum, 1/3rd of the maximum tension is absorbed as centrifugal tension.

**Notes :** 1. We know that $T_1 = T - T_C$ and for maximum power, $T_C = \frac{T}{3}$.

$$\therefore \quad T_1 = T - \frac{T}{3} = \frac{2T}{3}$$

2. From equation (*iv*), the velocity of the belt for the maximum power,

$$v = \sqrt{\frac{T}{3m}}$$

**Initial Tension in the Belt:**

When a belt is wound round the two pulleys (i.e. driver and follower), its two ends are joined together. In order to increase this grip, the belt is tightened up. At this stage, even when the pulleys are stationary, the belt is subjected to some tension, called initial tension.

When the driver starts rotating, it pulls the belt from one side (increasing tension in the belt on this side) and delivers it to the other side (decreasing the tension in the belt on that side). The increased tension in one side of the belt is called tension in tight side and the decreased tension in the other side of the belt is called tension in the slack side.

Let $T_0$ = Initial tension in the belt,

$T_1$= Tension in the tight side of the belt,

$T_2$= Tension in the slack side of the belt, and

$\alpha$ = Coefficient of increase of the belt length per unit force.

A little consideration will show that the increase of tension in the tight side

$$= T_1 - T_0$$

and increase in the length of the belt on the tight side

$$= \alpha (T_1 - T_0) \quad \text{...(i)}$$

Similarly, decrease in tension in the slack side

$$= T_0 - T_2$$

and decrease in the length of the belt on the slack side

$$= \alpha (T_0 - T_2) \quad \text{...(ii)}$$

Assuming that the belt material is perfectly elastic such that the length of the belt remains constant, when it is at rest or in motion, therefore increase in length on the tight side is equal to decrease in the length on the slack side. Thus, equating equations (i) and (ii),

$$\alpha (T_1 - T_0) = \alpha (T_0 - T_2) \text{ or } T_1 - T_0 = T_0 - T_2$$

$$\alpha (T_1 - T_0) = \alpha (T_0 - T_2) \text{ or } T_1 - T_0 = T_0 - T_2$$

$$\therefore \quad T_0 = \frac{T_1 + T_2}{2} \quad \text{...(Neglecting centrifugal tension)}$$

$$= \frac{T_1 + T_2 + 2T_C}{2} \quad \text{...(Considering centrifugal tension)}$$

**Example. 11.12.** *In a flat belt drive the initial tension is 2000 N. The coefficient of friction between the belt and the pulley is 0.3 and the angle of lap on the smaller pulley is 150°. The smaller pulley has a radius of 200 mm and rotates at 500 r.p.m. Find the power in kW transmitted by the belt.*

**Solution.** Given : $T_0 = 2000$ N ; $\mu_0 = 0.3$ ; $\theta = 150° = 150° \times \pi / 180 = 2.618$ rad ; $r_2 = 200$ mm or $d_2 = 400$ mm $= 0.4$ m ; $N_2 = 500$ r.p.m.

We know that velocity of the belt,

$$v = \frac{\pi d_2 . N_2}{60} = \frac{\pi \times 0.4 \times 500}{60} = 10.47 \text{ m/s}$$

Let $T_1$ = Tension in the tight side of the belt, and

$T_2$ = Tension in the slack side of the belt.

We know that initial tension ($T_0$),

$$2000 = \frac{T_1 + T_2}{2} \quad \text{or} \quad T_1 + T_2 = 4000 \text{ N} \qquad ...(i)$$

We also know that

$$2.3 \log\left(\frac{T_1}{T_2}\right) = \mu.\theta = 0.3 \times 2.618 = 0.7854$$

$$\log\left(\frac{T_1}{T_2}\right) = \frac{0.7854}{2.3} = 0.3415$$

or

$$\frac{T_1}{T_2} = 2.2 \quad ...(ii)$$

...(Taking antilog of 0.3415)

A military tank uses chain, belt and gear drives for its movement and operation.

From equations (*i*) and (*ii*),

$T_1 = 2750$ N ;

and $T_2 = 1250$ N

$\therefore$ Power transmitted, $P = (T_1 - T_2)\, v$

$= (2750 - 1250)\ 10.47$

$= 15\ 700$ W $= 15.7$ kW **Ans.**

**Example 11.13.** *Two parallel shafts whose centre lines are 4.8 m apart, are connected by open belt drive. The diameter of the larger pulley is 1.5 m and that of smaller pulley 1 m. The initial tension in the belt when stationary is 3 kN. The mass of the belt is 1.5 kg / m length. The coefficient of friction between the belt and the pulley is 0.3. Taking centrifugal tension into account, calculate the power transmitted, when the smaller pulley rotates at 400 r.p.m.*

**Solution.** Given : $x = 4.8$ m ; $d_1 = 1.5$ m ; $d_2 = 1$ m ; $T_0 = 3$ kN $= 3000$ N ; $m = 1.5$ kg / m ; $\mu = 0.3$ ; $N_2 = 400$ r.p.m.

We know that velocity of the belt,

$$v = \frac{\pi d_2 . N_2}{60} = \frac{\pi \times 1 \times 400}{60} = 21 \text{ m/s}$$

and centrifugal tension, $T_C = m.v^2 = 1.5\,(21)^2 = 661.5$ N

Let $T_1$ = Tension in the tight side, and

$T_2$ = Tension in the slack side.

We know that initial tension ($T_0$),

$$3000 = \frac{T_1 + T_2 + 2T_C}{2} = \frac{T_1 + T_2 + 2 \times 661.5}{2}$$

$$\therefore \quad T_1 + T_2 = 3000 \times 2 - 2 \times 661.5 = 4677 \text{ N} \qquad ...(i)$$

For an open belt drive,

$$\sin\alpha = \frac{r_1 - r_2}{x} = \frac{d_1 - d_2}{2x} = \frac{1.5 - 1}{2 \times 4.8} = 0.0521 \quad \text{or} \quad \alpha = 3°$$

$\therefore$ Angle of lap on the smaller pulley,

$$\theta = 180° - 2\,\alpha = 180° - 2 \times 3° = 174°$$

$$= 174° \times \pi / 180 = 3.04 \text{ rad}$$

We know that

$$2.3 \log\left(\frac{T_1}{T_2}\right) = \mu.\theta = 0.3 \times 3.04 = 0.912$$

$$\log\left(\frac{T_1}{T_2}\right) = \frac{0.912}{2.3} = 0.3965 \quad \text{or} \quad \frac{T_1}{T_2} = 2.5 \qquad ...(ii)$$

...(Taking antilog of 0.3965)

From equations (*i*) and (*ii*),

$T_1 = 3341$ N ; and $T_2 = 1336$ N

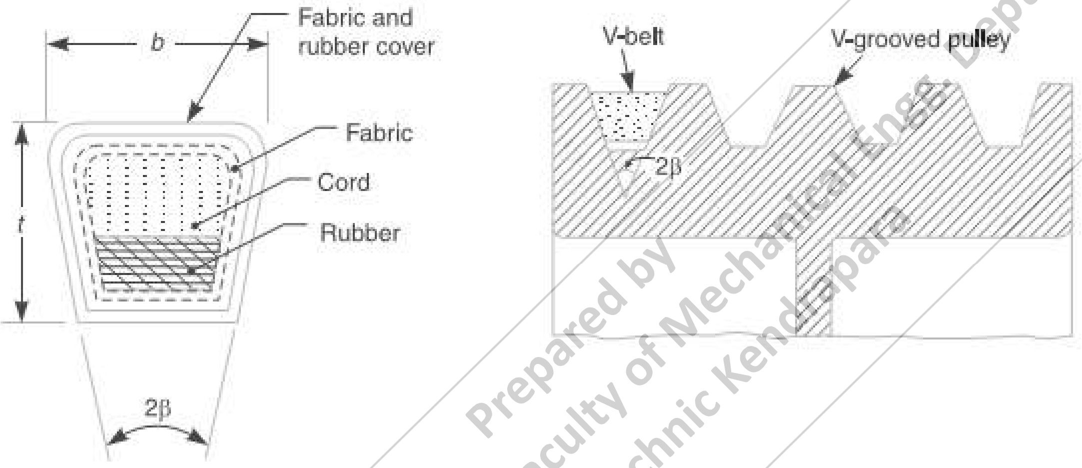
$\therefore$ Power transmitted,

$$P = (T_1 - T_2)\, v = (3341 - 1336)\ 21 = 42\ 100 \text{ W} = 42.1 \text{ kW} \ \textbf{Ans.}$$

**V-belt drive:**

A V-belt is mostly used in factories and workshops where a great amount of power is to be transmitted from one pulley to another when the two pulleys are very near to each other.

The V-belts are made of fabric and cords moulded in rubber and covered with fabric and rubber. These belts are moulded to a trapezoidal shape and are made endless. The included angle for the V-belt is usually from 30° – 40°. In case of V-belt drive, the rim of the pulley is grooved in which the V-belt runs. The effect of the groove is to increase the frictional grip of the V-belt on the pulley and thus to reduce the tendency of slipping. In order to have a good grip on the pulley, the V-belt is in contact with the side faces of the groove and not at the bottom. The power is transmitted by the wedging action between the belt and the V-groove in the pulley




(*a*) Cross-section of a V-belt. (*b*) Cross-section of a V-grooved pulley.

Fig. 11.19. V-belt and V-grooved pulley.

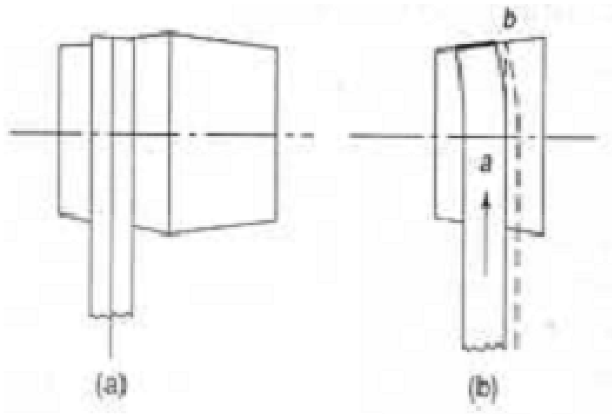


A clearance must be provided at the bottom of the groove, in order to prevent touching to the bottom as it becomes narrower from wear. The V-belt drive, may be inclined at any angle with tight side either at top or bottom. In order to increase the power output, several V- belts may be operated side by side. It may be noted that in multiple V-belt drive, all the belts should stretch at the same rate so that the load is equally divided between them. When one of the set of belts break, the entire set should be replaced at the same time. If only one belt is replaced, the new unworn and unstressed belt will be more tightly stretched and will move with different velocity.

**Crowning of pulleys:**

The rim of the pulley of a flat-belt drive is slightly crowned to prevent the slipping off belt from the pulley. The crowing can be in the from of conical surface or a convex surface.

Assume that somehow a belt comes over the conical portion of the pulley and takes the position as shown in figure, its centre line remains in a plane, the belt will touch the rim surface at its one edge only. This is impractical. Owing to the pull, the belt always tends to stick to the rim surface. The belt also has a lateral stiffness. Thus, a belt has to bend in the way shown in figure.



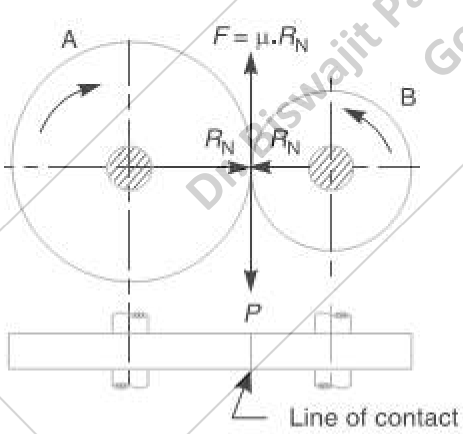

Let the belt travel in the direction of the arrow. As the belt touches the cone, the point a on it tends to adhere to the cone surface due to pull on the belt. This means as the pulley will take a quarter turn, the point a on the belt will be carried to b which is towards the mid-plane of the pulley than that previously occupied by the edge of the belt. But again, the belt cannot be stable on the pulley in the upright position and has to bend to stick to the cone surface; it will occupy the position shown by dotted lines.

Thus, if a pulley is made up of two equal cones or of convex surface, the belt will tend to climb on the slopes and will thus, run with its centre line on the mid-plane of the pulley. The amount of crowing is usually 1/96 of the pulley face width.

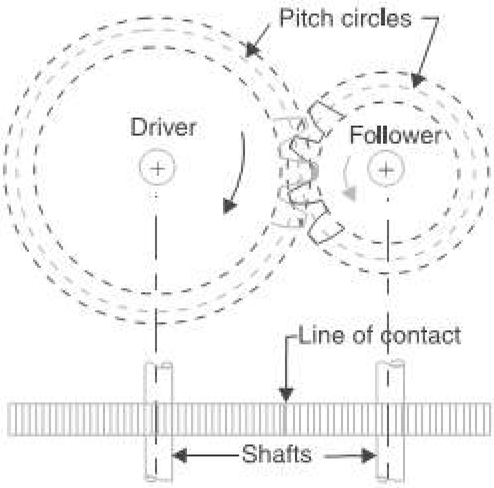


**Gear drives:**

In precision machines, in which a definite velocity ratio is of importance (as in watch mechanism), the only positive drive is by means of *gears* or *toothed wheels*.



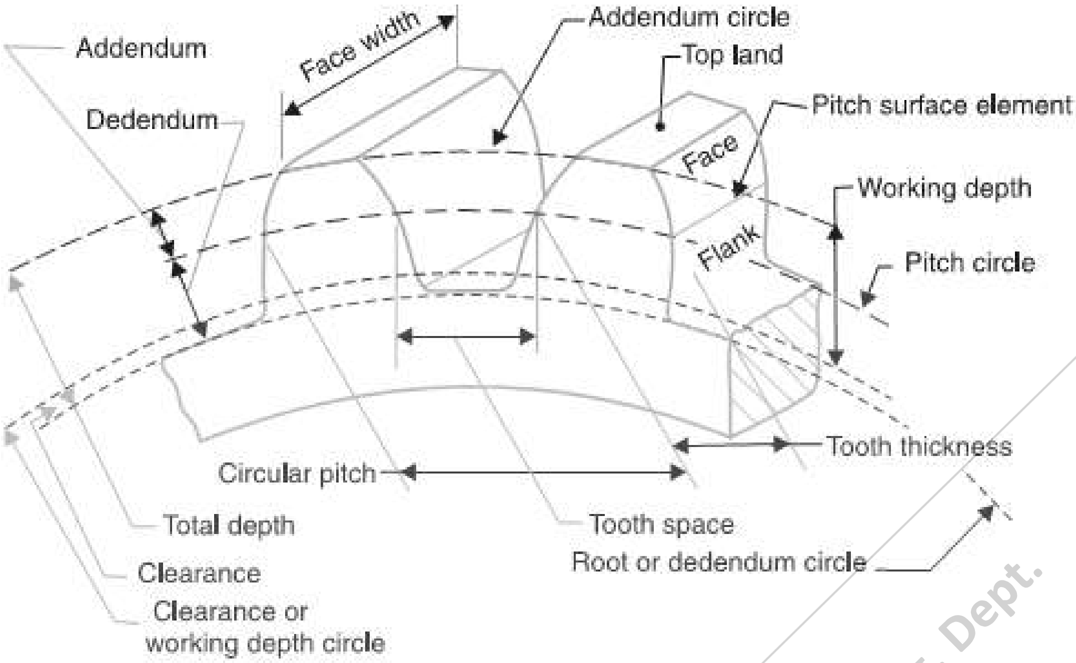

(*a*) Friction wheels.




(*b*) Toothed wheels.

In order to avoid the slipping, a number of projections (called teeth), are provided on the periphery of the wheel A, which will fit into the corresponding recesses on the periphery of the wheel B. A friction wheel with the teeth cut on it is known as toothed wheel or gear.

**Gear terminology:**




1. ***Pitch circle:*** It is an imaginary circle which by pure rolling action, would give the same motion as the actual gear.
2. ***Pitch circle diameter:*** It is the diameter of the pitch circle. The size of the gear is usually specified by the pitch circle diameter.
3. ***Pitch point:*** It is a common point of contact between two pitch circles.
4. ***Pitch surface:*** It is the surface of the rolling discs which the meshing gears have replaced at the pitch circle.
5. ***Pressure angle or angle of obliquity:*** It is the angle between the common normal to two gear teeth at the point of contact and the common tangent at the pitch point. It is usually denoted by φ. The standard pressure angles are 14 1/2 ° and 20°.
6. ***Addendum:*** It is the radial distance of a tooth from the pitch circle to the top of the tooth.
7. ***Dedendum:*** It is the radial distance of a tooth from the pitch circle to the bottom of the tooth.
8. ***Addendum circle:*** It is the circle drawn through the top of the teeth and is concentric with the pitch circle.
9. ***Dedendum circle:*** It is the circle drawn through the bottom of the teeth. It is also called root circle.
10. ***Circular pitch:*** It is the distance measured on the circumference of the pitch circle from a point of one tooth to the corresponding point on the next tooth. It is usually denoted by $p_c$.

Mathematically,

Circular pitch, $p_c = \pi D/T$

where `

D = Diameter of the pitch circle, and

T = Number of teeth on the wheel.

If $D_1$ and $D_2$ are the diameters of the two meshing gears having the teeth $T_1$ and $T_2$ respectively, then for them to mesh correctly,

$$p_c = \frac{\pi D_1}{T_1} = \frac{\pi D_2}{T_2} \quad \text{or} \quad \frac{D_1}{D_2} = \frac{T_1}{T_2}$$

***11. Diametral pitch:*** It is the ratio of number of teeth to the pitch circle diameter in millimeters. It is denoted by $p_d$.

Mathematically,

Diametral pitch, $p_d = T / D = \pi / p_c$

where ` D = Diameter of the pitch circle, and

T = Number of teeth on the wheel.

***12. Module:*** It is the ratio of the pitch circle diameter in millimeters to the number of teeth. It is usually denoted by m. Mathematically,

Module, $m = D / T$.

***13. Face of tooth:*** It is the surface of the gear tooth above the pitch surface.

***14. Flank of tooth:*** It is the surface of the gear tooth below the pitch surface.

***15. Face width:*** It is the width of the gear tooth measured parallel to its axis.

**Gear train:**

- When two or more gears are made to mesh with each other to transmit power from one shaft to another the combination is called gear train or train of toothed wheels.
- The nature of the train used depends upon the velocity ratio required and the relative position of the axes of shafts.
- A gear train may consist of spur, bevel or spiral gears.

**Types of Gear trains:**

1. Simple gear train,
2. Compound gear train,
3. Reverted gear train, and
4. Epicyclic gear train.

**1. Simple gear train:**

When there is only one gear on each shaft, it is known as simple gear train. The gears are represented by their pitch circles.



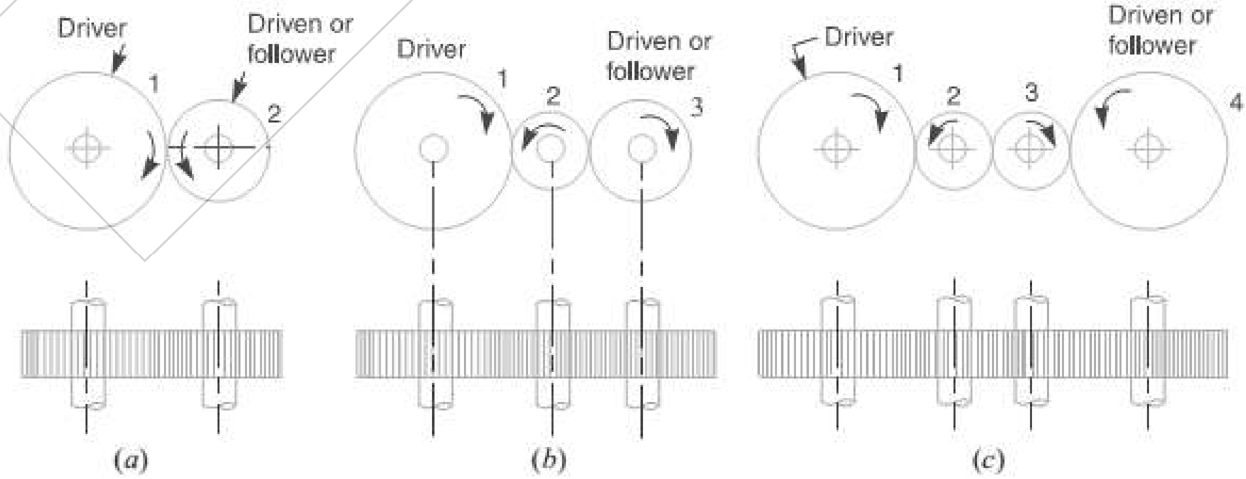



**Fig. 13.1.** Simple gear train.

Let $N_1$ & $N_2$ = Speed of gear 1 & gear 2 respectively in r.p.m.,

$T_1$ & $T_2$ = Number of teeth on gear 1, and on gear 2 respectively.

Since the speed ratio (or velocity ratio) of gear train is the ratio of the speed of the driver to the speed of the driven or follower and ratio of speeds of any pair of gears in mesh is the inverse of their number of teeth, therefore

$$\text{Speed ratio} = N_1 / N_2 = T_2 / T_1$$

It may be noted that ratio of the speed of the driven or follower to the speed of the driver is known as train value of the gear train.

Mathematically, $\text{Train value} = N_2 / N_1 = T_1 / T_2$

From above, we see that the *train value is the reciprocal of speed ratio*.

Sometimes, the distance between the two gears is large. The motion from one gear to another, in such a case, may be transmitted by either of the following two methods:

1. By providing the large sized gear, or 2. By providing one or more intermediate gears.

It may be noted that when the number of intermediate gears are odd, the motion of both the gears (i.e. driver and driven or follower) is like. But if the numbers of intermediate gears are even, the motion of the driven or follower will be in the opposite direction of the driver.

Now consider a simple train of gears with one intermediate gear.

Let $N_1$, $N_2$ & $N_3$ = Speed of driver, intermediate gear & driven respectively in r.p.m.,

$T_1$, $T_2$ & $T_3$ = Number of teeth on driver, intermediate gear & driven respectively.

Since the driving gear 1 is in mesh with the intermediate gear 2, therefore speed ratio for these two gears is

$$N_1 / N_2 = T_2 / T_1 \qquad \text{...(i)}$$

Similarly, as the intermediate gear 2 is in mesh with the driven gear 3, therefore speed ratio for these two gears is

$$N_2 / N_3 = T_3 / T_2 \qquad \text{...(ii)}$$

The speed ratio of the gear train is obtained by multiplying the equations (i) and (ii).

$$\therefore \quad \frac{N_1}{N_2} \times \frac{N_2}{N_3} = \frac{T_2}{T_1} \times \frac{T_3}{T_2} \quad \text{or} \quad \frac{N_1}{N_3} = \frac{T_3}{T_1}$$

*i.e.* $$\text{Speed ratio} = \frac{\text{Speed of driver}}{\text{Speed of driven}} = \frac{\text{No. of teeth on driven}}{\text{No. of teeth on driver}}$$

and $$\text{Train value} = \frac{\text{Speed of driven}}{\text{Speed of driver}} = \frac{\text{No. of teeth on driver}}{\text{No. of teeth on driven}}$$

From above, we see that the speed ratio and the train value, in a simple train of gears, is independent of the size and number of intermediate gears. These intermediate gears are called *idle gears*, as they do not effect the speed ratio or train value of the system. The idle gears are used for the following two purposes:

**a)** To connect gears where a large centre distance is required, and

**b)** To obtain the desired direction of motion of the driven gear (i.e. clockwise or anticlockwise).

**2. Compound gear train:**

When there are more than one gear on a shaft, it is called a ***compound train of gear***. These gears are useful in bridging over the space between the driver and the driven. In a compound train of gears, the gear 1 is the driving gear mounted on shaft A , gears 2 and 3 are compound gears which are mounted on shaft B. The gears 4 and 5 are also compound gears which are mounted on shaft C and the gear 6 is the driven gear mounted on shaft D.

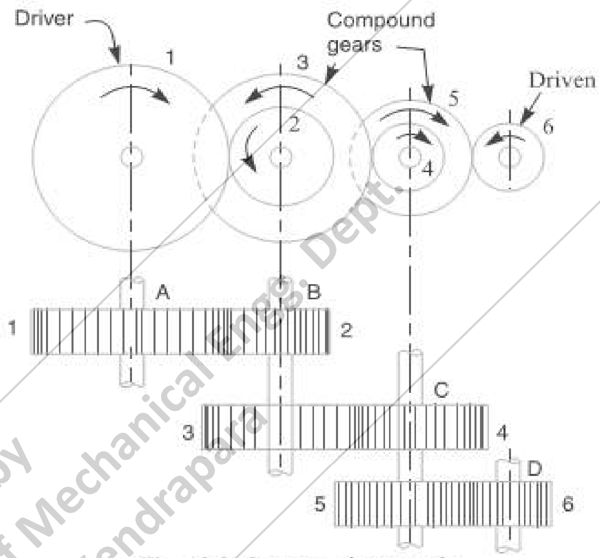



Fig. 13.2. Compound gear train.



Let $N_1$ = Speed of driving gear 1, $T_1$ = Number of teeth on driving gear 1,

$N_2, N_3 ..., N_6$ = Speed of respective gears in r.p.m., and

$T_2, T_3..., T_6$ = Number of teeth on respective gears.

Since gear 1 is in mesh with gear 2, therefore its speed ratio is

$$N_1 / N_2 = T_2 / T_1 \quad ...(i)$$

Similarly, for gears 3 and 4, speed ratio is

$$N_3 / N_4 = T_4 / T_3 \quad ...(ii)$$

and for gears 5 and 6, speed ratio is

$$N_5 / N_6 = T_6 / T_5 \quad ...(ii)$$

The speed ratio of compound gear train is obtained by multiplying the equations (i), (ii) and (iii),

$$\therefore \quad \frac{N_1}{N_2} \times \frac{N_3}{N_4} \times \frac{N_5}{N_6} = \frac{T_2}{T_1} \times \frac{T_4}{T_3} \times \frac{T_6}{T_5} \quad \text{or} \quad {}^*\frac{N_1}{N_6} = \frac{T_2 \times T_4 \times T_6}{T_1 \times T_3 \times T_5}$$

*i.e.*

$$\text{Speed ratio} = \frac{\text{Speed of the first driver}}{\text{Speed of the last driven or follower}} = \frac{\text{Product of the number of teeth on the drivens}}{\text{Product of the number of teeth on the drivers}}$$

and

$$\text{Train value} = \frac{\text{Speed of the last driven or follower}}{\text{Speed of the first driver}} = \frac{\text{Product of the number of teeth on the drivers}}{\text{Product of the number of teeth on the drivens}}$$

The advantage of a compound train over a simple gear train is that a much larger speed reduction from the first shaft to the last shaft can be obtained with small gears. If a simple gear train is used to give a large speed reduction, the last gear has to be very large. Usually for a speed reduction in excess of 7 to 1, a simple train is not used and a compound train or worm gearing is employed.

**Note:** The gears which mesh must have the same circular pitch or module. Thus gears 1 and 2 must have the same module as they mesh together. Similarly gears 3 and 4, and gears 5 and 6 must have the same module.

**Example 13.1.** *The gearing of a machine tool is shown in Fig. 13.3. The motor shaft is connected to gear A and rotates at 975 r.p.m. The gear wheels B, C, D and E are fixed to parallel shafts rotating together. The final gear F is fixed on the output shaft. What is the speed of gear F ? The number of teeth on each gear are as given below :*

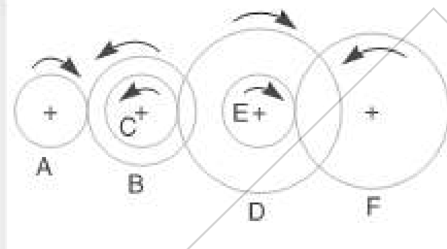



Fig. 13.3

| *Gear* | *A* | *B* | *C* | *D* | *E* | *F* |
|---|---|---|---|---|---|---|
| *No. of teeth* | *20* | *50* | *25* | *75* | *26* | *65* |

**Solution.** Given : $N_A = 975$ r.p.m. ; $T_A = 20$ ; $T_B = 50$ ; $T_C = 25$ ; $T_D = 75$ ; $T_E = 26$ ; $T_F = 65$

From Fig. 13.3, we see that gears *A*, *C* and *E* are drivers while the gears *B*, *D* and *F* are driven or followers. Let the gear *A* rotates in clockwise direction. Since the gears *B* and *C* are mounted on the same shaft, therefore it is a compound gear and the direction or rotation of both these gears is same (*i.e.* anticlockwise). Similarly, the gears *D* and *E* are mounted on the same shaft, therefore it is also a compound gear and the direction of rotation of both these gears is same (*i.e.* clockwise). The gear *F* will rotate in anticlockwise direction.

Battery Car: Even though it is run by batteries, the power transmission, gears, clutches, brakes, etc. remain mechanical in nature.

Note : This picture is given as additional information and is not a direct example of the current chapter.

Let $N_F$ = Speed of gear *F*, *i.e.* last driven or follower.

We know that

$$\frac{\text{Speed of the first driver}}{\text{Speed of the last driven}} = \frac{\text{Product of no. of teeth on drivens}}{\text{Product of no. of teeth on drivers}}$$

or

$$\frac{N_A}{N_F} = \frac{T_B \times T_D \times T_F}{T_A \times T_C \times T_E} = \frac{50 \times 75 \times 65}{20 \times 25 \times 26} = 18.75$$

$$\therefore \quad N_F = \frac{N_A}{18.75} = \frac{975}{18.75} = 52 \text{ r. p. m.} \quad \textbf{Ans.}$$

**3. Reverted gear train:**

When the axes of the first gear (i.e. first driver) and the last gear (i.e. last driven or follower) are co-axial, then the gear train is known as *reverted gear train*.

We see that gear 1 (i.e. first driver) drives the gear 2 (i.e. first driven or follower) in the opposite direction. Since the gears 2 and 3 are mounted on the same shaft, therefore they form a compound gear and the gear 3 will rotate in the same direction as that of gear 2. The gear 3 (which is now the second driver) drives the gear 4 (i.e. the last driven or follower) in the same direction as that of gear 1. Thus we see that in a reverted gear train, the motion of the first gear and the last gear is *like*.

Let $T_1$ = Number of teeth on gear 1,

$r_1$ = Pitch circle radius of gear 1, and

$N_1$ = Speed of gear 1 in r.p.m.

Similarly,

$T_2$, $T_3$, $T_4$ = Number of teeth on respective gears,

$r_2$, $r_3$, $r_4$ = Pitch circle radii of respective gears, and

$N_2$, $N_3$, $N_4$ = Speed of respective gears in r.p.m.

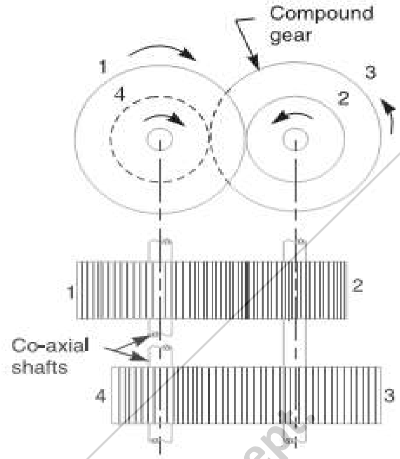


Fig. 13.4. Reverted gear train.

Since the distance between the centres of the shafts of gears 1 and 2 as well as gears 3 and 4 is same, therefore

$$r_1 + r_2 = r_3 + r_4 \qquad ...(i)$$

Also, the circular pitch or module of all the gears is assumed to be same, therefore number of teeth on each gear is directly proportional to its circumference or radius.

$$\therefore \qquad T_1 + T_2 = T_3 + T_4 \qquad ...(ii)$$

and

$$\text{Speed ratio} = \frac{\text{Product of number of teeth on drivens}}{\text{Product of number of teeth on drivers}}$$

or

$$\frac{N_1}{N_4} = \frac{T_2 \times T_4}{T_1 \times T_3} \qquad ... (iii)$$

From equations *(i)*, *(ii)* and *(iii)*, we can determine the number of teeth on each gear for the given centre distance, speed ratio and module only when the number of teeth on one gear is chosen arbitrarily.

The reverted gear trains are used in automotive transmissions, lathe back gears, industrial speed reducers, and in clocks (where the minute and hour hand shafts are co-axial).

**4. Epicyclic gear train:**

We have already discussed that in an epicyclic gear train, the axes of the shafts, over which the gears are mounted, may move relative to a fixed axis. A simple epicyclic gear train, where a gear A and the arm C have a common axis at $O_1$ about which they can rotate. The gear B meshes with gear A and has its axis on the arm at $O_2$, about which the gear B can rotate.

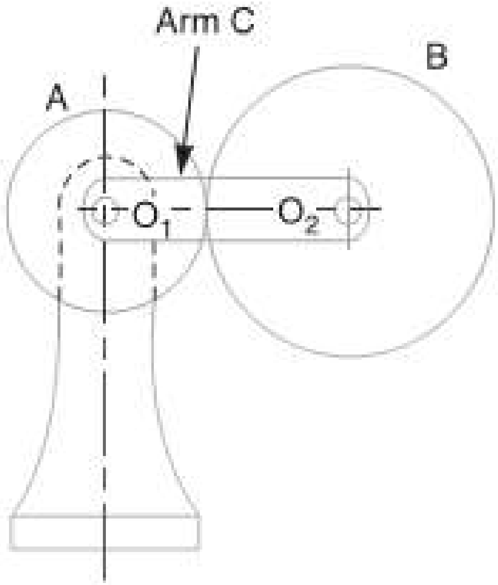


Fig. 13.6. Epicyclic gear train.

If the arm is fixed, the gear train is simple and gear A can drive gear B or vice- versa, but if gear A is fixed and the arm is rotated about the axis of gear A (i.e. $O_1$), then the gear B is forced to rotate upon and around gear A. Such a motion is called *epicyclic* and the gear trains arranged in such a manner that one or more of their members move upon and around another member is known as *epicyclic gear trains* (*epi.* means upon and *cyclic* means around). The epicyclic gear trains may be *simple* or *compound*.

The epicyclic gear trains are useful for transmitting high velocity ratios with gears of moderate size in a comparatively lesser space. The epicyclic gear trains are used in the back gear of lathe, differential gears of the automobiles, hoists, pulley blocks, wrist watches etc.



Table 13.1. Table of motions

| *Step No.* | *Conditions of motion* | *Revolutions of elements* | | |
|---|---|---|---|---|
| | | *Arm C* | *Gear A* | *Gear B* |
| 1. | Arm fixed-gear *A* rotates through + 1 revolution *i.e.* 1 rev. anticlockwise | 0 | + 1 | $-\frac{T_A}{T_B}$ |
| 2. | Arm fixed-gear *A* rotates through + *x* revolutions | 0 | + *x* | $-x \times \frac{T_A}{T_B}$ |
| 3. | Add + *y* revolutions to all elements | + *y* | + *y* | + *y* |
| 4. | Total motion | + *y* | *x* + *y* | $y - x \times \frac{T_A}{T_B}$ |

# Chapter-04: Governors and Flywheel

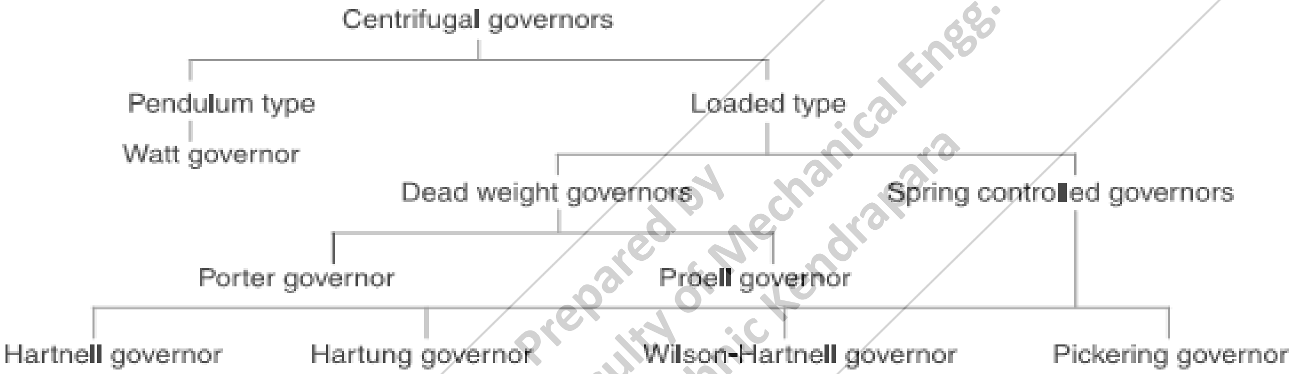
## Governors and its functions:

- The function of a governor is to regulate the mean speed of an engine, when there are variations in the load.
- When the load on an engine increases, its speed decreases, therefore it becomes necessary to increase the supply of working fluid. On the other hand, when the load on the engine decreases, its speed increases and thus less working fluid is required.
- The governor automatically controls the supply of working fluid to the engine with the varying load conditions and keeps the mean speed within certain limits.

## Classification of Governors:

The governors may, broadly, be classified as

1. Centrifugal governors, and 2. Inertia governors.






## 1) Watt Governor:

It is basically a conical pendulum with links attached to a sleeve of negligible mass. The arms of the governor may be connected to the spindle in the following three ways:
**1.** The pivot *P*, may be on the spindle axis as shown in Fig. 18.2 (*a*).
**2.** The pivot *P*, may be offset from the spindle axis and the arms when produced intersect at *O*, as shown in Fig. 18.2 (*b*).
**3.** The pivot *P*, may be offset, but the arms cross the axis at *O*, as shown in Fig. 18.2 (*c*).

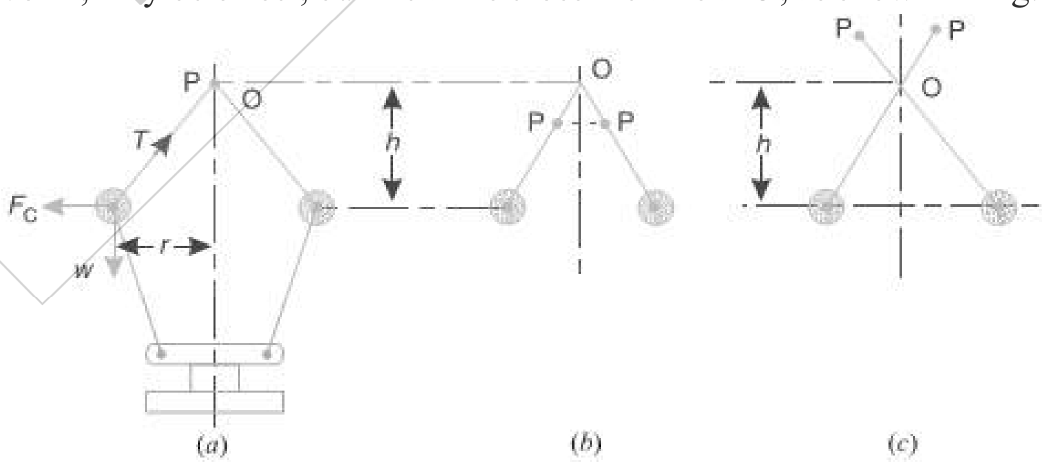



**Fig. 18.2.** Watt governor.

Let m = Mass of the ball in kg,

w = Weight of the ball in newtons = m.g,

T = Tension in the arm in newtons,

ω = Angular velocity of the arm and ball about the spindle axis in rad/s,

r = Radius of the path of rotation of the ball i.e. horizontal distance from the centre of the ball to the spindle axis in metres,

$F_C$= Centrifugal force acting on the ball in newtons = m. ω 2.r, and

h = Height of the governor in metres.

It is assumed that the weight of the arms, links and the sleeve are negligible as compared to the weight of the balls. Now, the ball is in equilibrium under the action of

1. the centrifugal force (FC) acting on the ball, 2. the tension (T) in the arm, and 3. the weight (w) of the ball.

Taking moments about point O, we have

$$F_C \times h = w \times r = m.g.r$$

$$\text{or} \quad m.\omega^2.r.h = m.g.r \quad \text{or} \quad h = g / \omega^2 \qquad \ldots (i)$$

When g is expressed in $m/s^2$ and ω in rad/s, then h is in metres. If N is the speed in r.p.m., then

$$\omega = 2\pi N/60$$

$$\therefore \quad h = \frac{9.81}{(2\pi N/60)^2} = \frac{895}{N^2} \text{ metres} \qquad \ldots (\because g = 9.81 \text{ m/s}^2) \ldots (ii)$$

**Example 18.1**. *Calculate the vertical height of a Watt governor when it rotates at 60 r.p.m. Also find the change in vertical height when its speed increases to 61 r.p.m.*

**Solution.** Given : $N_1$ = 60 r.p.m. ; $N_2$ = 61 r.p.m.

***Initial height***

We know that initial height,

$$h_1 = \frac{895}{(N_1)^2} = \frac{895}{(60)^2} = 0.248 \text{ m}$$

***Change in vertical height***

We know that final height,

$$h_2 = \frac{895}{(N_2)^2} = \frac{895}{(61)^2} = 0.24 \text{ m}$$

∴ Change in vertical height

$$= h_1 - h_2 = 0.248 - 0.24 = 0.008 \text{ m} = 8 \text{ mm } \textbf{Ans.}$$

**2) Porter Governor*:***

The Porter governor is a modification of a Watt's governor, with central load attached to the sleeve. The load moves up and down the central spindle. This additional downward force increases the speed of revolution required to enable the balls to rise to any predetermined level. Consider the forces acting on one-half of the governor as shown in Fig. 18.3 (*b*).

Let m = Mass of each ball in kg,

w = Weight of each ball in newtons = m.g

M = Mass of the central load in kg,

W = Weight of the central load in newtons = M.g,
r = Radius of rotation in metres,
h = Height of governor in metres ,
N = Speed of the balls in r.p.m .,
ω = Angular speed of the balls in rad/s = 2 π N/60 rad/s,
$F_C$ = Centrifugal force acting on the ball in newtons = m. ω $^2$.r,
$T_1$= Force in the arm in newtons,
$T_2$= Force in the link in newtons,
α = Angle of inclination of the arm (or upper link) to the vertical, and
β = Angle of inclination of the link (or lower link) to the vertical.

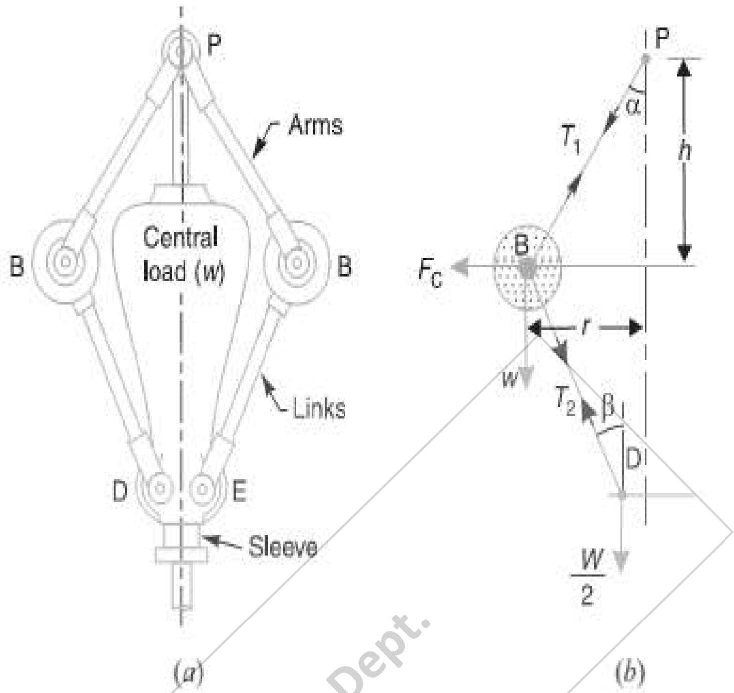



Fig. 18.3. Porter governor.



Though there are several ways of determining the relation between the height of the governor (h) and the angular speed of the balls (ω), yet the following two methods are important from the subject point of view:

1. Method of resolution of forces; and 2. Instantaneous centre method.

***1. Method of resolution of forces***

Considering the equilibrium of the forces acting at D, we have

$$T_2 \cos\beta = \frac{W}{2} = \frac{M.g}{2}$$

or

$$T_2 = \frac{M.g}{2\cos\beta} \qquad \ldots (i)$$

Again, considering the equilibrium of the forces acting on B. The point B is in equilibrium under the action of the following forces, as shown in Fig. 18.3 (b).

*(i)* The weight of ball (w = m.g),
*(ii)* The centrifugal force ($F_C$),
*(iii)* The tension in the arm ($T_1$), and
*(iv)* The tension in the link ($T_2$).

Resolving the forces vertically

$$T_1 \cos\alpha = T_2 \cos\beta + w = \frac{M.g}{2} + m.g \qquad \ldots (ii)$$

$$\ldots \left( \because \; T_2 \cos\beta = \frac{M.g}{2} \right)$$

Resolving the forces horizontally,

$$T_1 \sin\alpha + T_2 \sin\beta = F_C$$

$$T_1 \sin\alpha + \frac{M.g}{2\cos\beta} \times \sin\beta = F_C \qquad \ldots \left( \because \; T_2 = \frac{M.g}{2\cos\beta} \right)$$

$$T_1 \sin\alpha + \frac{M.g}{2} \times \tan\beta = F_C$$

$$\therefore \quad T_1 \sin\alpha = F_C - \frac{M.g}{2} \times \tan\beta \qquad \ldots (iii)$$

Dividing equation (*iii*) by equation (*ii*),

$$\frac{T_1 \sin\alpha}{T_1 \cos\alpha} = \frac{F_C - \dfrac{M.g}{2} \times \tan\beta}{\dfrac{M.g}{2} + m.g}$$

or
$$\left(\frac{M.g}{2} + m.g\right) \tan\alpha = F_C - \frac{M.g}{2} \times \tan\beta$$

$$\frac{M.g}{2} + m.g = \frac{F_C}{\tan\alpha} - \frac{M.g}{2} \times \frac{\tan\beta}{\tan\alpha}$$

Substituting $\dfrac{\tan\beta}{\tan\alpha} = q$, and $\tan\alpha = \dfrac{r}{h}$, we have

$$\frac{M.g}{2} + m.g = m.\omega^2.r \times \frac{h}{r} - \frac{M.g}{2} \times q \qquad \ldots (\therefore F_C = m.\omega^2.r)$$

or
$$m.\omega^2.h = m.g + \frac{M.g}{2}(1 + q)$$

$\therefore$
$$h = \left[m.g + \frac{M.g}{2}(1+q)\right]\frac{1}{m.\omega^2} = \frac{m + \dfrac{M}{2}(1+q)}{m} \times \frac{g}{\omega^2} \qquad \ldots (iv)$$

or
$$\omega^2 = \left[m.g + \frac{Mg}{2}(1+q)\right]\frac{1}{m.h} = \frac{m + \dfrac{M}{2}(1+q)}{m} \times \frac{g}{h}$$

or
$$\left(\frac{2\pi N}{60}\right)^2 = \frac{m + \dfrac{M}{2}(1+q)}{m} \times \frac{g}{h}$$

$\therefore$
$$N^2 = \frac{m + \dfrac{M}{2}(1+q)}{m} \times \frac{g}{h}\left(\frac{60}{2\pi}\right)^2 = \frac{m + \dfrac{M}{2}(1+q)}{m} \times \frac{895}{h} \qquad \ldots (v)$$

. . . (Taking $g = 9.81$ m/s²)

**Notes : 1.** When the length of arms are equal to the length of links and the points *P* and *D* lie on the same vertical line, then

$$\tan\alpha = \tan\beta \quad \text{or} \quad q = \tan\alpha / \tan\beta = 1$$

Therefore, the equation (*v*) becomes

$$N^2 = \frac{(m + M)}{m} \times \frac{895}{h} \qquad \ldots (vi)$$

**2.** When the loaded sleeve moves up and down the spindle, the frictional force acts on it in a direction opposite to that of the motion of sleeve.

If *F* = Frictional force acting on the sleeve in newtons, then the equations (*v*) and (*vi*) may be written as

$$N^2 = \frac{m.g + \left(\dfrac{M.g \pm F}{2}\right)(1+q)}{m.g} \times \frac{895}{h} \qquad \ldots (vii)$$

$$= \frac{m.g + (M.g \pm F)}{m.g} \times \frac{895}{h} \qquad \ldots (\text{When } q = 1) \ldots (viii)$$

The + sign is used when the sleeve moves upwards or the governor speed increases and negative sign is used when the sleeve moves downwards or the governor speed decreases.

3. On comparing the equation (*vi*) with equation (*ii*) of Watt's governor (Art. 18.5), we find that the mass of the central load ($M$) increases the height of governor in the ratio $\frac{m+M}{m}$.

**2. *Instantaneous centre method***

In this method, equilibrium of the forces acting on the link $BD$ are considered. The instantaneous centre $I$ lies at the point of intersection of $PB$ produced and a line through $D$ perpendicular to the spindle axis, as shown in Fig. 18.4. Taking moments about the point $I$,

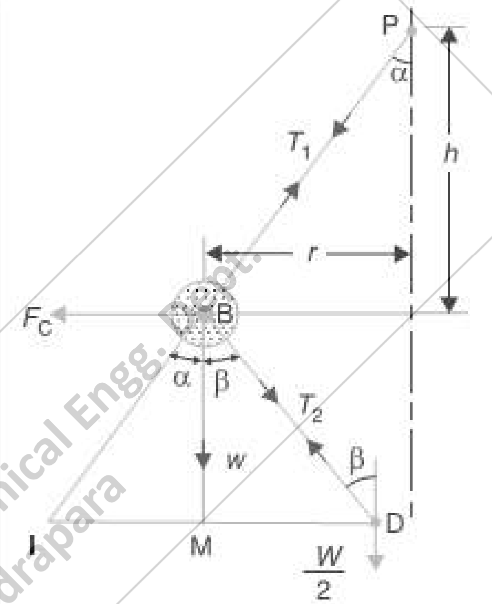



Fig. 18.4. Instantaneous centre method.

$$F_C \times BM = w \times IM + \frac{W}{2} \times ID$$

$$= m.g \times IM + \frac{M.g}{2} \times ID$$

$$\therefore \quad F_C = m.g \times \frac{IM}{BM} + \frac{M.g}{2} \times \frac{ID}{BM}$$

$$= m.g \times \frac{IM}{BM} + \frac{M.g}{2}\left(\frac{IM + MD}{BM}\right)$$

$$= m.g \times \frac{IM}{BM} + \frac{M.g}{2}\left(\frac{IM}{BM} + \frac{MD}{BM}\right)$$

$$= m.g \tan\alpha + \frac{M.g}{2}(\tan\alpha + \tan\beta)$$

$$\ldots\left(\because \frac{IM}{BM} = \tan\alpha, \text{ and } \frac{MD}{BM} = \tan\beta\right)$$

Dividing throughout by $\tan\alpha$,

$$\frac{F_C}{\tan\alpha} = m.g + \frac{M.g}{2}\left(1 + \frac{\tan\beta}{\tan\alpha}\right) = m.g + \frac{M.g}{2}(1+q) \qquad \ldots\left(\because q = \frac{\tan\beta}{\tan\alpha}\right)$$

We know that $F_C = m.\omega^2.r$, and $\tan\alpha = \frac{r}{h}$

$$\therefore \quad m.\omega^2.r \times \frac{h}{r} = m.g + \frac{M.g}{2}(1+q)$$

or

$$h = \frac{m.g + \frac{M.g}{2}(1+q)}{m} \times \frac{1}{\omega^2} = \frac{m + \frac{M}{2}(1+q)}{m} \times \frac{g}{\omega^2}$$

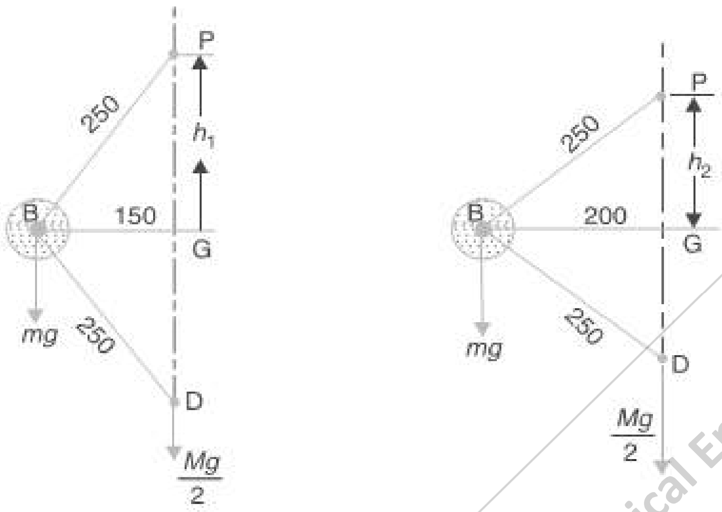
. . . (Same as before)

When $\tan\alpha = \tan\beta$ or $q = 1$, then

$$h = \frac{m+M}{m} \times \frac{g}{\omega^2}$$

**Example 18.2.** *A Porter governor has equal arms each 250 mm long and pivoted on the axis of rotation. Each ball has a mass of 5 kg and the mass of the central load on the sleeve is 25 kg. The radius of rotation of the ball is 150 mm when the governor begins to lift and 200 mm when the governor is at maximum speed. Find the minimum and maximum speeds and range of speed of the governor.*

**Solution.** Given : $BP = BD = 250$ mm $= 0.25$ m ; $m = 5$ kg ; $M = 15$ kg ; $r_1 = 150$ mm $= 0.15$m; $r_2 = 200$ mm $= 0.2$ m



(*a*) Minimum position. (*b*) Maximum position.

**Fig. 18.5**



The minimum and maximum positions of the governor are shown in Fig. 18.5 (*a*) and (*b*) respectively.

***Minimum speed when $r_1 = BG = 0.15$ m***

Let $N_1$ = Minimum speed.

From Fig. 18.5 (*a*), we find that height of the governor,

$$h_1 = PG = \sqrt{(PB)^2 - (BG)^2} = \sqrt{(0.25)^2 - (0.15)^2} = 0.2 \text{ m}$$

We know that

$$(N_1)^2 = \frac{m + M}{m} \times \frac{895}{h_1} = \frac{5 + 15}{5} \times \frac{895}{0.2} = 17\,900$$

$\therefore$ $N_1 = 133.8$ r.p.m. **Ans.**

***Maximum speed when $r_2 = BG = 0.2$ m***

Let $N_2$ = Maximum speed.

From Fig. 18.5 (*b*), we find that height of the governor,

$$h_2 = PG = \sqrt{(PB)^2 - (BG)^2} = \sqrt{(0.25)^2 - (0.2)^2} = 0.15 \text{ m}$$

We know that

$$(N_2)^2 = \frac{m+M}{m} \times \frac{895}{h_2} = \frac{5+15}{5} \times \frac{895}{0.15} = 23\,867$$

$\therefore$ $N_2 = 154.5$ r.p.m. **Ans.**

***Range of speed***

We know that range of speed

$$= N_2 - N_1 = 154.4 - 133.8 = 20.7 \text{ r.p.m. } \textbf{Ans.}$$

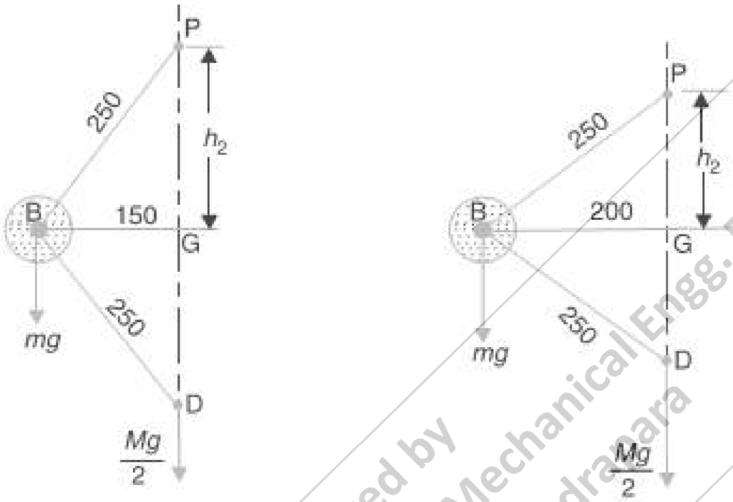
**Example 18.3.** *The arms of a Porter governor are each 250 mm long and pivoted on the governor axis. The mass of each ball is 5 kg and the mass of the central sleeve is 30 kg. The radius of rotation of the balls is 150 mm when the sleeve begins to rise and reaches a value of 200 mm for maximum speed. Determine the speed range of the governor. If the friction at the sleeve is equivalent of 20 N of load at the sleeve, determine how the speed range is modified.*

**Solution.** Given : $BP = BD = 250$ mm ; $m = 5$ kg ; $M = 30$ kg ; $r_1 = 150$ mm ; $r_2 = 200$ mm

First of all, let us find the minimum and maximum speed of the governor. The minimum and maximum position of the governor is shown in Fig. 18.6 (*a*) and (*b*) respectively.

Let $N_1$ = Minimum speed when $r_1 = BG = 150$ mm, and

$N_2$ = Maximum speed when $r_2 = BG = 200$ mm.




(*a*) Minimum position. (*b*) Maximum position.

**Fig. 18.6**

*Speed range of the governor*

From Fig. 18.6 (*a*), we find that height of the governor,

$$h_1 = PG = \sqrt{(PB)^2 - (BG)^2} = \sqrt{(250)^2 - (150)^2} = 200 \text{ mm} = 0.2 \text{ m}$$

We know that

$$(N_1)^2 = \frac{m + M}{m} \times \frac{895}{h_1} = \frac{5 + 30}{5} \times \frac{895}{0.2} = 31\,325$$

$\therefore$ $N_1$ = 177 r.p.m.

From Fig. 18.6 (*b*), we find that height of the governor,

$$h_2 = PG = \sqrt{(PB)^2 - (BG)^2} = \sqrt{(250)^2 - (200)^2} = 150 \text{ mm} = 0.15 \text{ m}$$

We know that

$$(N_2)^2 = \frac{m + M}{m} \times \frac{895}{h_2} = \frac{5 + 30}{5} \times \frac{895}{0.15} = 41\,767$$

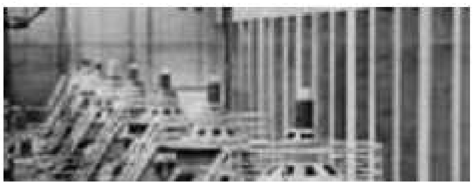
$\therefore$ $N_2$ = 204.4 r.p.m.

We know that speed range of the governor

$= N_2 - N_1 = 204.4 - 177 = 27.4$ r.p.m. **Ans.**

***Speed range when friction at the sleeve is equivalent of 20 N of load (i.e. when F = 20 N)***

We know that when the sleeve moves downwards, the friction force (*F*) acts upwards and the minimum speed is given by



$$(N_1)^2 = \frac{m.g + (M.g - F)}{m.g} \times \frac{895}{h_1}$$

$$= \frac{5 \times 9.81 + (30 \times 9.81 - 20)}{5 \times 9.81} \times \frac{895}{0.2} = 29500$$

$\therefore \qquad N_1 = 172$ r.p.m.

We also know that when the sleeve moves upwards, the frictional force ($F$) acts downwards and the maximum speed is given by

$$(N_2)^2 = \frac{m.g + (M.g + F)}{m.g} \times \frac{895}{h_2}$$

$$= \frac{5 \times 9.81 + (30 \times 9.81 + 20)}{5 \times 9.81} \times \frac{895}{0.15} = 44200$$

$\therefore \qquad N_2 = 210$ r.p.m.

We know that speed range of the governor

$= N_2 - N_1 = 210 - 172 = 38$ r.p.m. **Ans.**

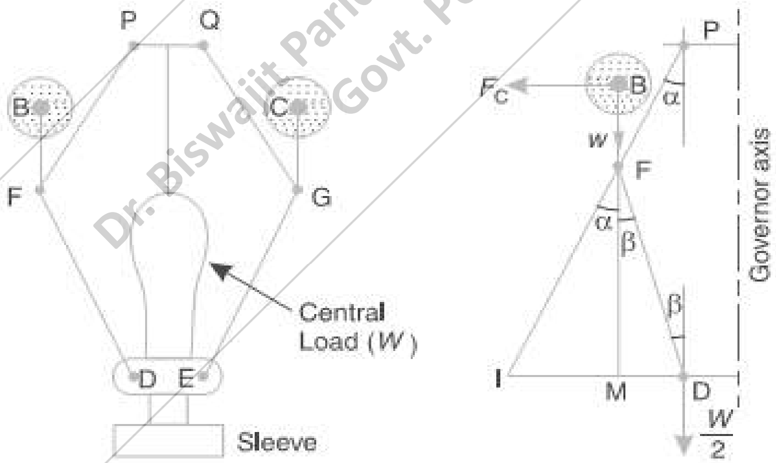
A series of hydel generators.

Note : This picture is given as additional information and is not a direct example of the current chapter.

### 3) Proell Governor:

The Proell governor has the balls fixed at *B* and *C* to the extension of the links *DF* and *EG*, as shown in Fig. 18.12 (*a*). The arms *FP* and *GQ* are pivoted at *P* and *Q* respectively.

Consider the equilibrium of the forces on one-half of the governor as shown in Fig. 18.12 (*b*). The instantaneous centre (*I*) lies on the intersection of the line *PF* produced and the line from *D* drawn perpendicular to the spindle axis. The perpendicular *BM* is drawn on *ID*.




(*a*) (*b*)

**Fig. 18.12.** Proell governor.

Taking moments about *I*, using the same notations as discussed in Art. 18.6 (Porter governor),

$$F_C \times BM = w \times IM + \frac{W}{2} \times ID = m.g \times IM + \frac{M.g}{2} \times ID \qquad \ldots (i)$$

$$\therefore \qquad F_C = m.g \times \frac{IM}{BM} + \frac{M.g}{2}\left(\frac{IM + MD}{BM}\right) \qquad \ldots (\because \; ID = IM + MD)$$

Multiplying and dividing by $FM$, we have

$$F_C = \frac{FM}{BM}\left[m.g \times \frac{IM}{FM} + \frac{M.g}{2}\left(\frac{IM}{FM} + \frac{MD}{FM}\right)\right]$$

$$= \frac{FM}{BM}\left[m.g \times \tan\alpha + \frac{M.g}{2}(\tan\alpha + \tan\beta)\right]$$

$$= \frac{FM}{BM} \times \tan\alpha\left[m.g + \frac{M.g}{2}\left(1 + \frac{\tan\beta}{\tan\alpha}\right)\right]$$

We know that $F_C = m.\omega^2 r$; $\tan\alpha = \frac{r}{h}$ and $q = \frac{\tan\beta}{\tan\alpha}$

$$\therefore \quad m.\omega^2.r = \frac{FM}{BM} \times \frac{r}{h}\left[m.g + \frac{M.g}{2}(1+q)\right]$$

and
$$\omega^2 = \frac{FM}{BM}\left[\frac{m + \frac{M}{2}(1+q)}{m}\right]\frac{g}{h} \quad \ldots (ii)$$

Substituting $\omega = 2\pi N/60$, and $g = 9.81$ m/s$^2$, we get

$$N^2 = \frac{FM}{BM}\left[\frac{m + \frac{M}{2}(1+q)}{m}\right]\frac{895}{h} \quad \ldots (iii)$$

**Notes : 1.** The equation (*i*) may be applied to any given configuration of the governor.

**2.** Comparing equation (*iii*) with the equation (*v*) of the Porter governor (Art. 18.6), we see that the equilibrium speed reduces for the given values of *m*, *M* and *h*. Hence in order to have the same equilibrium speed for the given values of *m*, *M* and *h*, balls of smaller masses are used in the Proell governor than in the Porter governor.

**3.** When $\alpha = \beta$, then $q = 1$. Therefore equation (*iii*) may be written as

$$N^2 = \frac{FM}{BM}\left(\frac{m+M}{m}\right)\frac{895}{h} \quad (h \text{ being in metres}) \ldots (iv)$$



**Example 18.9.** *A Proell governor has equal arms of length 300 mm. The upper and lower ends of the arms are pivoted on the axis of the governor. The extension arms of the lower links are each 80 mm long and parallel to the axis when the radii of rotation of the balls are 150 mm and 200 mm. The mass of each ball is 10 kg and the mass of the central load is 100 kg. Determine the range of speed of the governor.*

**Solution.** Given : $PF = DF = 300$ mm ; $BF = 80$ mm ; $m = 10$ kg ; $M = 100$ kg ; $r_1 = 150$ mm; $r_2 = 200$ mm

First of all, let us find the minimum and maximum speed of the governor. The minimum and maximum position of the governor is shown in Fig. 18.13.

Let $N_1$ = Minimum speed when radius of rotation, $r_1 = FG = 150$ mm ; and

$N_2$ = Maximum speed when radius of rotation, $r_2 = FG = 200$ mm.

From Fig. 18.13 (*a*), we find that height of the governor,

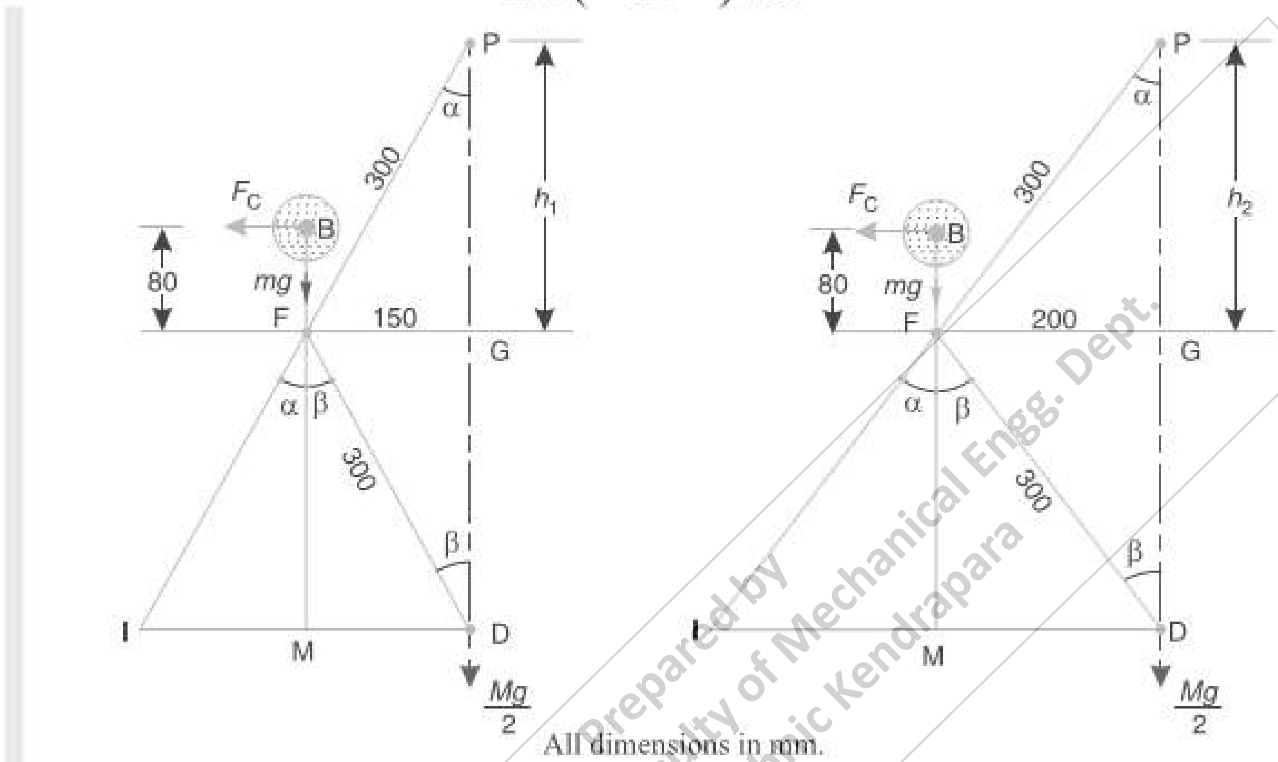
$$h_1 = PG = \sqrt{(PF)^2 - (FG)^2} = \sqrt{(300)^2 - (150)^2} = 260 \text{ mm} = 0.26 \text{ m}$$

and $FM = GD = PG = 260 \text{ mm} = 0.26 \text{ m}$

$\therefore$ $BM = BF + FM = 80 + 260 = 340 \text{ mm} = 0.34 \text{ m}$

We know that $$(N_1)^2 = \frac{FM}{BM}\left(\frac{m+M}{m}\right)\frac{895}{h_1} \quad \ldots (\therefore\ \alpha = \beta \text{ or } q = 1)$$

$$= \frac{0.26}{0.34}\left(\frac{10+100}{10}\right)\frac{895}{0.26} = 28\,956 \quad \text{or} \quad N_1 = 170 \text{ r.p.m.}$$




(*a*) Minimum position. (*a*) Maximum position.

**Fig. 18.13**

Now from Fig. 18.13 (*b*), we find that height of the governor,

$$h_2 = PG = \sqrt{(PF)^2 - (FG)^2} = \sqrt{(300)^2 - (200)^2} = 224 \text{ mm} = 0.224 \text{ m}$$

and $FM = GD = PG = 224 \text{ mm} = 0.224 \text{ m}$

$\therefore$ $BM = BF + FM = 80 + 224 = 304 \text{ mm} = 0.304 \text{ m}$

We know that $$(N_2)^2 = \frac{FM}{BM}\left(\frac{m+M}{m}\right)\frac{895}{h_2} \quad \ldots (\because\ \alpha = \beta \text{ or } q = 1)$$

$$= \frac{0.224}{0.304}\left(\frac{10+100}{10}\right)\frac{895}{0.224} = 32\,385 \quad \text{or} \quad N_2 = 180 \text{ r.p.m.}$$

We know that range of speed

$$= N_2 - N_1 = 180 - 170 = 10 \text{ r.p.m. } \textbf{Ans.}$$

**Note :** The example may also be solved as discussed below :

From Fig. 18.13 (*a*), we find that

$$\sin\alpha = \sin\beta = 150 / 300 = 0.5 \quad \text{or} \quad \alpha = \beta = 30°$$

and $MD = FG = 150 \text{ mm} = 0.15 \text{ m}$

$FM = FD\cos\beta = 300\cos 30° = 260 \text{ mm} = 0.26 \text{ m}$

$$IM = FM \tan\alpha = 0.26 \tan 30° = 0.15 \text{ m}$$
$$BM = BF + FM = 80 + 260 = 340 \text{ mm} = 0.34 \text{ m}$$
$$ID = IM + MD = 0.15 + 0.15 = 0.3 \text{ m}$$

We know that centrifugal force,

$$F_C = m\,(\omega_1)^2 r_1 = 10\left(\frac{2\pi N_1}{60}\right)^2 0.15 = 0.0165\,(N_1)^2$$

Now taking moments about point *I*,

$$F_C \times BM = m.g \times IM + \frac{M.g}{2} \times ID$$

or $$0.0165\,(N_1)^2\; 0.34 = 10 \times 9.81 \times 0.15 + \frac{100 \times 9.81}{2} \times 0.3$$

$$0.0056\,(N_1)^2 = 14.715 + 147.15 = 161.865$$

$$\therefore \quad (N_1)^2 = \frac{161.865}{0.0056} = 28\,904 \quad \text{or} \quad N_1 = 170 \text{ r.p.m.}$$

Similarly $N_2$ may be calculated.

**Example 18.10.** *A governor of the Proell type has each arm 250 mm long. The pivots of the upper and lower arms are 25 mm from the axis. The central load acting on the sleeve has a mass of 25 kg and the each rotating ball has a mass of 3.2 kg. When the governor sleeve is in mid-position, the extension link of the lower arm is vertical and the radius of the path of rotation of the masses is 175 mm. The vertical height of the governor is 200 mm.*

*If the governor speed is 160 r.p.m. when in mid-position, find : 1. length of the extension link; and 2. tension in the upper arm.*

**Solution.** Given : $PF = DF = 250$ mm $= 0.25$ m ; $PQ = DH = KG = 25$ mm $= 0.025$ m $M = 25$ kg ; $m = 3.2$ kg ; $r = FG = 175$ mm $= 0.175$ m ; $h = QG = PK = 200$ mm $= 0.2$ m ; $N = 160$ r.p.m

**1.** ***Length of the extension link***

Let $BF$ = Length of the extension link.

The Proell governor in its mid-position is shown in Fig. 18.14.

From the figure, we find that

$$FM = GH = QG = 200 \text{ mm} = 0.2 \text{ m}$$

We know that

$$N^2 = \frac{FM}{BM}\left(\frac{m + M}{m}\right)\frac{895}{h} \quad \ldots (\because\ \alpha = \beta \text{ or } q = 1)$$

$$(160)^2 = \frac{0.2}{BM}\left(\frac{3.2 + 25}{3.2}\right)\frac{895}{0.2} = \frac{7887}{BM}$$

$$\therefore \quad BM = 7887/(160)^2 = 0.308 \text{ m}$$

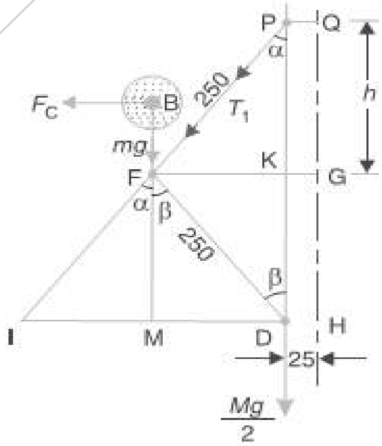


**Fig. 18.14.** All dimensions in mm.

From Fig. 18.14,

$$BF = BM - FM = 0.308 - 0.2 = 0.108 \text{ m} = 108 \text{ mm} \quad \textbf{Ans.}$$

**2.** ***Tension in the upper arm***

Let $T_1$ = Tension in the upper arm.

$$PK = \sqrt{(PF)^2 - (FK)^2} = \sqrt{(PF)^2 - (FG - KG)^2}$$

$$= \sqrt{(250)^2 - (175 - 25)^2} = 200 \text{ mm}$$

$$\cos\alpha = PK/PF = 200/250 = 0.8$$

and $$T_1 \cos\alpha = mg + \frac{Mg}{2} = 3.2 \times 9.81 + \frac{25 \times 9.81}{2} = 154 \text{ N}$$

$$\therefore \quad T_1 = \frac{154}{\cos\alpha} = \frac{154}{0.8} = 192.5 \text{ N} \quad \textbf{Ans.}$$

**4) Hartnell Governor*:***

A Hartnell governor is a spring loaded governor. It consists of two bell crank levers pivoted at the points O,O to the frame. The frame is attached to the governor spindle and therefore rotates with it. Each lever carries a ball at the end of the vertical arm OB and a roller at the end of the horizontal arm OR. A helical spring in compression provides equal downward forces on the two rollers through a collar on the sleeve. The spring force may be adjusted by screwing a nut up or down on the sleeve.

Let $m$ = Mass of each ball in kg,

$M$ = Mass of sleeve in kg,

$r_1$ = Minimum radius of rotation in metres,

$r_2$ = Maximum radius of rotation in metres,

$\omega_1$ = Angular speed of the governor at minimum radius in rad/s,

$\omega_2$ = Angular speed of the governor at maximum radius in rad/s,

$S_1$ = Spring force exerted on the sleeve at $\omega_1$ in newtons,

$S_2$ = Spring force exerted on the sleeve at $\omega_2$ in newtons,

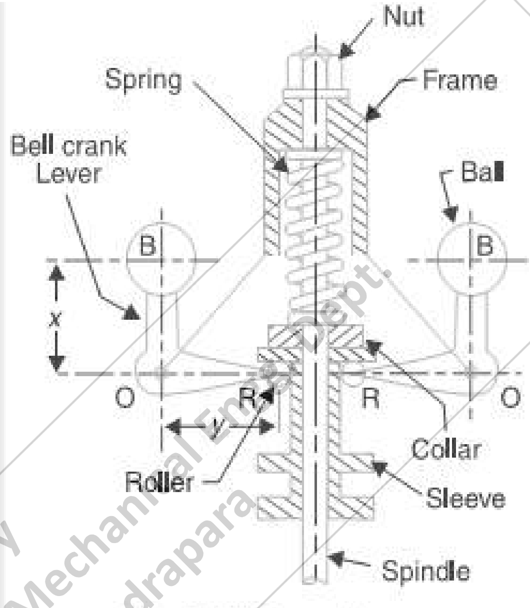



**Fig. 18.18.** Hartnell governor.

$F_{C1}$ = Centrifugal force at $\omega_1$ in newtons = $m(\omega_1)^2 r_1$,

$F_{C2}$ = Centrifugal force at $\omega_2$ in newtons = $m(\omega_2)^2 r_2$,

s = Stiffness of the spring or the force required to compress the spring by one mm,

x = Length of the vertical or ball arm of the lever in metres,

y = Length of the horizontal or sleeve arm of the lever in metres, and

r = Distance of fulcrum O from the governor axis or the radius of rotation when the governor is in mid-position, in metres.

Consider the forces acting at one bell crank lever. The minimum and maximum position is shown in Fig. 18.19. Let h be the compression of the spring when the radius of rotation changes from $r_1$ to $r_2$.

For the minimum position i.e. when the radius of rotation changes from r to $r_1$, as shown in Fig. 18.19 (a), the compression of the spring or the lift of sleeve $h_1$ is given by

$$\frac{h_1}{y} = \frac{a_1}{x} = \frac{r - r_1}{x} \quad \ldots (i)$$

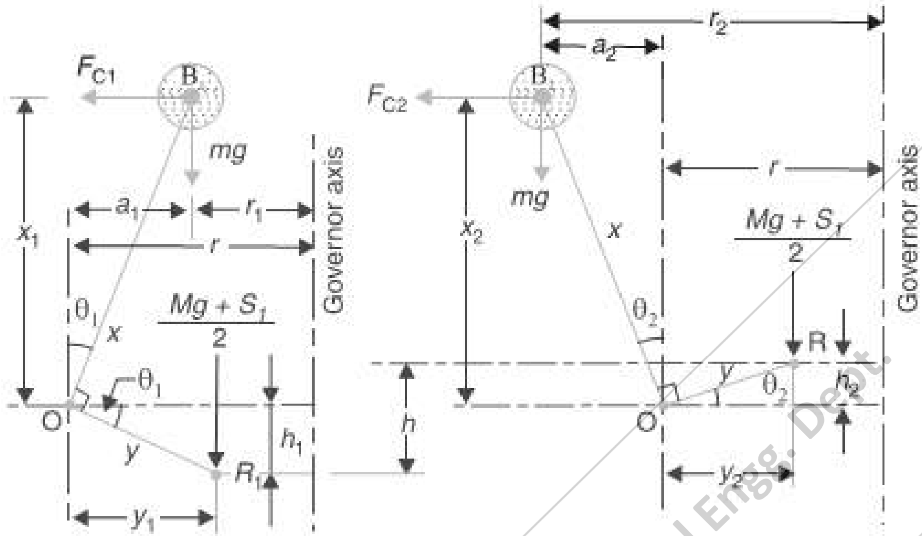
Similarly, for the maximum position *i.e.* when the radius of rotation changes from $r$ to $r_2$, as shown in Fig. 18.19 (*b*), the compression of the spring or lift of sleeve $h_2$ is given by

$$\frac{h_2}{y} = \frac{a_2}{x} = \frac{r_2 - r}{x} \quad \ldots (ii)$$

Adding equations (*i*) and (*ii*),

$$\frac{h_1 + h_2}{y} = \frac{r_2 - r_1}{x} \quad \text{or} \quad \frac{h}{y} = \frac{r_2 - r_1}{x} \qquad \ldots (\because\ h = h_1 + h_2)$$

$$\therefore \qquad h = (r_2 - r_1)\,\frac{y}{x} \qquad \ldots (iii)$$




(*a*) Minimum position. (*b*) Maximum position.

**Fig. 18.19**

Now for minimum position, taking moments about point *O*, we get

$$\frac{M.g + S_1}{2} \times y_1 = F_{C1} \times x_1 - m.g \times a_1$$

or
$$M.g + S_1 = \frac{2}{y_1}(F_{C1} \times x_1 - m.g \times a_1) \qquad \ldots (iv)$$

Again for maximum position, taking moments about point *O*, we get

$$\frac{M.g + S_2}{2} \times y_2 = F_{C2} \times x_2 + m.g \times a_2$$

or
$$M.g + S_2 = \frac{2}{y_2}(F_{C2} \times x_2 + m.g \times a_2) \qquad \ldots (v)$$

Subtracting equation (*iv*) from equation (*v*),

$$S_2 - S_1 = \frac{2}{y_2}(F_{C2} \times x_2 + m.g \times a_2) - \frac{2}{y_1}(F_{C1} \times x_1 - m.g \times a_1)$$

We know that

$$S_2 - S_1 = h.s, \qquad \text{and} \qquad h = (r_2 - r_1)\,\frac{y}{x}$$

$$\therefore \qquad s = \frac{S_2 - S_1}{h} = \left(\frac{S_2 - S_1}{r_2 - r_1}\right)\frac{x}{y}$$

Neglecting the obliquity effect of the arms (*i.e.* $x_1 = x_2 = x$, and $y_1 = y_2 = y$) and the moment due to weight of the balls (*i.e.* $m.g$), we have for minimum position,

$$\frac{M.g + S_1}{2} \times y = F_{C1} \times x \quad \text{or} \quad M.g + S_1 = 2F_{C1} \times \frac{x}{y} \qquad \ldots (vi)$$

Similarly for maximum position,

$$\frac{M.g + S_2}{2} \times y = F_{C2} \times x \quad \text{or} \quad M.g + S_2 = 2F_{C2} \times \frac{x}{y} \quad \ldots (vii)$$

Subtracting equation (*vi*) from equation (*vii*),

$$S_2 - S_1 = 2\,(F_{C2} - F_{C1})\,\frac{x}{y} \quad \ldots (viii)$$

We know that

$$S_2 - S_1 = h.s, \quad \text{and} \quad h = (r_2 - r_1)\,\frac{y}{x}$$

$$\therefore \quad s = \frac{S_2 - S_1}{h} = 2\left(\frac{F_{C2} - F_{C1}}{r_2 - r_1}\right)\left(\frac{x}{y}\right)^2 \quad \ldots (ix)$$

**Notes : 1.** Unless otherwise stated, the obliquity effect of the arms and the moment due to the weight of the balls is neglected, in actual practice.

**2.** When friction is taken into account, the weight of the sleeve ($M.g$) may be replaced by ($M.g. \pm F$).

**3.** The centrifugal force ($F_C$) for any intermediate position (*i.e.* between the minimum and maximum position) at a radius of rotation ($r$) may be obtained as discussed below :

Since the stiffness for a given spring is constant for all positions, therefore for minimum and intermediate position,

$$s = 2\left(\frac{F_C - F_{C1}}{r - r_1}\right)\left(\frac{x}{y}\right)^2 \quad \ldots (x)$$

and for intermediate and maximum position,

$$s = 2\left(\frac{F_{C2} - F_C}{r_2 - r}\right)\left(\frac{x}{y}\right)^2 \quad \ldots (xi)$$

$\therefore$ From equations (*ix*), (*x*) and (*xi*),

$$\frac{F_{C2} - F_{C1}}{r_2 - r_1} = \frac{F_C - F_{C1}}{r - r_1} = \frac{F_{C2} - F_C}{r_2 - r}$$

or

$$F_C = F_{C1} + (F_{C2} - F_{C1})\left(\frac{r - r_1}{r_2 - r_1}\right) = F_{C2} - (F_{C2} - F_{C1})\left(\frac{r_2 - r}{r_2 - r_1}\right)$$

**Example 18.13.** *A Hartnell governor having a central sleeve spring and two right-angled bell crank levers moves between 290 r.p.m. and 310 r.p.m. for a sleeve lift of 15 mm. The sleeve arms and the ball arms are 80 mm and 120 mm respectively. The levers are pivoted at 120 mm from the governor axis and mass of each ball is 2.5 kg. The ball arms are parallel to the governor axis at the lowest equilibrium speed. Determine : 1. loads on the spring at the lowest and the highest equilibrium speeds, and 2. stiffness of the spring.*

**Solution.** Given : $N_1 = 290$ r.p.m. or $\omega_1 = 2\,\pi \times 290/60 = 30.4$ rad/s ; $N_2 = 310$ r.p.m. or $\omega_2 = 2\,\pi \times 310/60 = 32.5$ rad/s ; $h = 15$ mm $= 0.015$ m ; $y = 80$ mm $= 0.08$ m ; $x = 120$ mm $= 0.12$ m ; $r = 120$ mm $= 0.12$ m ; $m = 2.5$ kg

**1.** ***Loads on the spring at the lowest and highest equilibrium speeds***

Let $S$ = Spring load at lowest equilibrium speed, and

$S_2$ = Spring load at highest equilibrium speed.

Since the ball arms are parallel to governor axis at the lowest equilibrium speed (*i.e.* at $N_1 = 290$ r.p.m.), as shown in Fig. 18.20 (*a*), therefore

$$r = r_1 = 120 \text{ mm} = 0.12 \text{ m}$$

We know that centrifugal force at the minimum speed,

$$F_{C1} = m\,(\omega_1)^2\,r_1 = 2.5\,(30.4)^2\;0.12 = 277 \text{ N}$$

Now let us find the radius of rotation at the highest equilibrium speed, *i.e.* at $N_2 = 310$ r.p.m. The position of ball arm and sleeve arm at the highest equilibrium speed is shown in Fig. 18.20 (*b*).

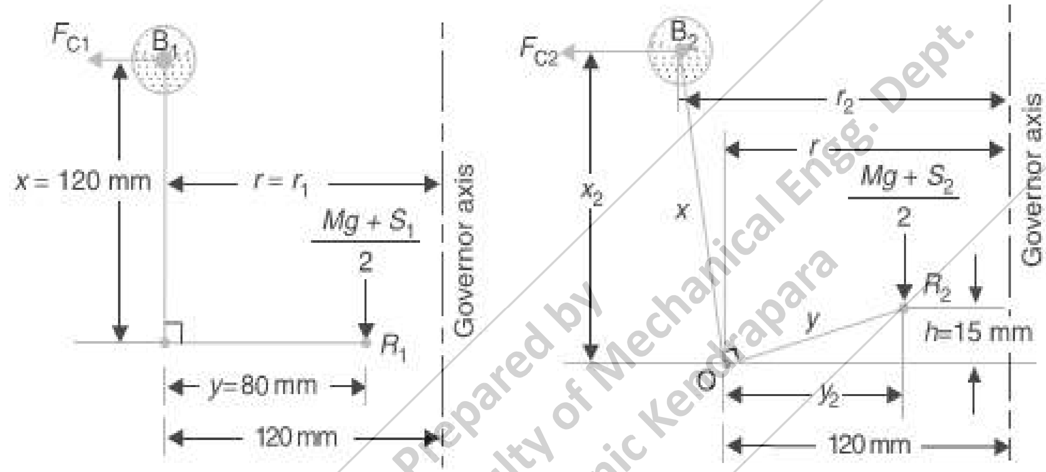
Let $\quad r_2$ = Radius of rotation at $N_2 = 310$ r.p.m.

We know that $\quad h = (r_2 - r_1)\dfrac{y}{x}$

or $$r_2 = r_1 + h\left(\frac{x}{y}\right) = 0.12 + 0.015\left(\frac{0.12}{0.08}\right) = 0.1425 \text{ m}$$

$\therefore$ Centrifugal force at the maximum speed,

$$F_{C2} = m\,(\omega_2)^2\,r_2 = 2.5 \times (32.5)^2 \times 0.1425 = 376 \text{ N}$$




(*a*) Lowest position. (*b*) Highest position.

**Fig. 18.20**



Neglecting the obliquity effect of arms and the moment due to the weight of the balls, we have for lowest position,

$$M.g + S_1 = 2F_{C1} \times \frac{x}{y} = 2 \times 277 \times \frac{0.12}{0.08} = 831 \text{ N}$$

$\therefore \quad S_2 = 831$ N **Ans.** $\quad (\because M = 0)$

and for highest position,

$$M.g + S_2 = 2F_{C2} \times \frac{x}{y} = 2 \times 376 \times \frac{0.12}{0.08} = 1128 \text{ N}$$

$\therefore \quad S_1 = 1128$ N **Ans.** $\quad (\because M = 0)$

**2.** ***Stiffness of the spring***

We know that stiffness of the spring,

$$s = \frac{S_2 - S_1}{h} = \frac{1128 - 831}{15} = 19.8 \text{ N/mm} \;\; \textbf{Ans.}$$

**Example 18.14.** *In a spring loaded Hartnell type governor, the extreme radii of rotation of the balls are 80 mm and 120 mm. The ball arm and the sleeve arm of the bell crank lever are equal in length. The mass of each ball is 2 kg. If the speeds at the two extreme positions are 400 and 420 r.p.m., find : 1. the initial compression of the central spring, and 2. the spring constant.*

**Solution.** Given : $r_1$ = 80 mm = 0.08 m ; $r_2$ = 120 mm = 0.12 m ; $x = y$ ; $m$ = 2 kg ; $N_1$ = 400 r.p.m. or $\omega = 2\pi \times 400/60 = 41.9$ rad/s ; $N_2$ = 420 r.p.m. or $\omega_2 = 2\pi \times 420/60 = 44$ rad/s

***Initial compression of the central spring***

We know that the centrifugal force at the minimum speed,

$$F_{C1} = m(\omega_1)^2 r_1 = 2(41.9)^2 \; 0.08 = 281 \text{ N}$$

and centrifugal force at the maximum speed,

$$F_{C2} = m(\omega_2)^2 r_2 = 2(44)^2 \; 0.12 = 465 \text{ N}$$

Let $S_1$ = Spring force at the minimum speed, and

$S_2$ = Spring force at the maximum speed.

We know that for minimum position,

$$M.g + S_1 = 2\,F_{C1} \times \frac{x}{y}$$

$$\therefore \quad S_1 = 2\,F_{C1} = 2 \times 281 = 562 \text{ N} \qquad \ldots (\because M = 0 \text{ and } x = y)$$

Similarly for maximum position,

$$M.g + S_2 = 2\,F_{C2} \times \frac{x}{y}$$

$$\therefore \quad S_2 = 2\,F_{C2} = 2 \times 465 = 930 \text{ N}$$

We know that lift of the sleeve,

$$h = (r_2 - r_1)\frac{y}{x} = r_2 - r_1 = 120 - 80 = 40 \text{ mm} \qquad \ldots (\because x = y)$$

$\therefore$ Stiffness of the spring,

$$s = \frac{S_2 - S_1}{h} = \frac{930 - 562}{40} = 9.2 \text{ N/mm}$$

We know that initial compression of the central spring

$$= \frac{S_1}{s} = \frac{562}{9.2} = 61 \text{ mm} \quad \textbf{Ans.}$$

**2.** ***Spring constant***

We have calculated above that the spring constant or stiffness of the spring,

$$s = 9.2 \text{ N/mm } \textbf{Ans.}$$

**Sensitiveness of Governors:**

Consider two governors A and B running at the same speed. When this speed increases or decreases by a certain amount, the lift of the sleeve of governor A is greater than the lift of the sleeve of governor B. It is then said that the governor A is more sensitive than the governor B.

In general, the greater the lift of the sleeve corresponding to a given fractional change in speed, the greater is the sensitiveness of the governor. It may also be stated in another way that for a given lift of the sleeve, the sensitiveness of the governor increases as the speed range decreases. When the governor is fitted to an engine, the practical requirement is simply that the change of equilibrium speed from the full load to the no load position of the sleeve should be as small a fraction as possible of the mean equilibrium speed. For this reason, *the sensitiveness is defined as the ratio of the difference between the maximum and minimum equilibrium speeds to the mean equilibrium speed*.

Let $N_1$ & $N_2$= Minimum and maximum equilibrium speed respectively,

$$N = \text{Mean equilibrium speed} = \frac{N_1 + N_2}{2}.$$

∴ Sensitiveness of the governor

$$= \frac{N_2 - N_1}{N} = \frac{2(N_2 - N_1)}{N_1 + N_2}$$

$$= \frac{2(\omega_2 - \omega_1)}{\omega_1 + \omega_2} \quad \text{. . . (In terms of angular speeds)}$$

**Stability of Governors*:***

A governor is said to be stable when for every speed within the working range there is a definite configuration i.e. there is only one radius of rotation of the governor balls at which the governor is in equilibrium. For a stable governor, if the equilibrium speed increases, the radius of governor balls must also increase.

*Note:* A governor is said to be unstable, if the radius of rotation decreases as the speed increases.

**Isochronous Governors*:***

A governor is said to be ***isochronous*** when the equilibrium speed is constant (*i.e.* range of speed is zero) for all radii of rotation of the balls within the working range, neglecting friction. The isochronism is the stage of infinite sensitivity.

Let us consider the case of a Porter governor running at speeds $N_1$ and $N_2$ r.p.m. We have,

$$(N_1)^2 = \frac{m + \frac{M}{2}(1+q)}{m} \times \frac{895}{h_1} \quad \text{. . . (i)}$$

and

$$(N_2)^2 = \frac{m + \frac{M}{2}(1+q)}{m} \times \frac{895}{h_2} \quad \text{. . . (ii)}$$

For isochronism, range of speed should be zero *i.e.* $N_2 - N_1 = 0$ or $N_2 = N_1$. Therefore from equations ***(i)*** and ***(ii)***, $h_1 = h_2$, which is impossible in case of a Porter governor. Hence a ***Porter governor cannot be isochronous.*** Now consider the case of a Hartnell governor running at speeds $N_1$ and $N_2$ r.p.m. We have discussed that

$$M.g + S_1 = 2\,F_{C1} \times \frac{x}{y} = 2 \times m \left(\frac{2\pi N_1}{60}\right)^2 r_1 \times \frac{x}{y} \quad \text{. . . (iii)}$$

and

$$M.g + S_2 = 2\,F_{C2} \times \frac{x}{y} = 2 \times m \left(\frac{2\pi N_2}{60}\right)^2 r_2 \times \frac{x}{y} \quad \text{. . . (iv)}$$

For isochronism, $N_2 = N_1$. Therefore from equations ***(iii)*** and ***(iv)***,

$$\frac{M.g + S_1}{M.g + S_2} = \frac{r_1}{r_2}$$

**Flywheel and its functions:**

- A flywheel used in machines serves as a reservoir, which stores energy during the period when the supply of energy is more than the requirement, and releases it during the period when the requirement of energy is more than the supply.
- In case of steam engines, internal combustion engines, reciprocating compressors and pumps, the energy is developed during one stroke and the engine is to run for the whole cycle on the energy produced during this one stroke.
- When the flywheel absorbs energy, its speed increases and when it releases energy, the speed decreases. Hence a flywheel does not maintain a constant speed, it simply reduces the fluctuation of speed.
- A flywheel controls the speed variations caused by the fluctuation of the engine turning moment during each cycle of operation.
- In machines where the operation is intermittent like *punching machines, shearing machines, riveting machines, crushers, etc., the flywheel stores energy from the power source during the greater portion of the operating cycle and gives it up during a small period of the cycle.
- Thus, the energy from the power source to the machines is supplied practically at a constant rate throughout the operation.

**Comparison between Flywheel & Governor:**

| Governor | Flywheel |
|---|---|
| • Its function is to regulate the supply of driving fluid producing energy, according to the load requirements so that at different loads almost a constant speed is maintained. | • Its function is to store available mechanical energy when it is in excess of the load requirement and provide the same when the available energy is less than that required by the load. |
| • It is provided on prime movers such as engines and turbines. | • It is provided on engine and fabricating machines e.g., rolling mills, punching machines shear machines, presses, etc. |
| • It takes care of fluctuation of speed due to variation of load over long range of working of engines and turbines. | • In engines it takes care of fluctuations of speed during thermodynamic cycle. |
| • It works intermittently i.e. only when there is change in the load. | • It works continuously from cycle to cycle. |
| • But for governor, there would have been unnecessarily more consumption of driving fluid. Thus it economises its consumptions. | • In fabrication machines, it is very economical to use it as its use reduces capital investment on prime movers and their running expenses. |

**Coefficient of Fluctuation of Speed:**

The difference between the maximum and minimum speeds during a cycle is called the maximum fluctuation of speed. The ratio of the maximum fluctuation of speed to the mean speed is called the coefficient of fluctuation of speed.

Let $N_1$ and $N_2$ = Maximum and minimum speeds in r.p.m. during the cycle, and

$N$ = Mean speed in r.p.m. = $(N_1 + N_2)/2$

∴ Coefficient of fluctuation of speed,

$$C_S = \frac{N_1 - N_2}{N} = \frac{2(N_1 - N_2)}{N_1 + N_2}$$

$$= \frac{\omega_1 - \omega_2}{\omega} = \frac{2(\omega_1 - \omega_2)}{\omega_1 + \omega_2} \quad \text{...(In terms of angular speeds)}$$

$$= \frac{v_1 - v_2}{v} = \frac{2(v_1 - v_2)}{v_1 + v_2} \quad \text{...(In terms of linear speeds)}$$

The coefficient of fluctuation of speed is a limiting factor in the design of flywheel. It varies depending upon the nature of service to which the flywheel is employed.

**Coefficient of Steadiness:**

The reciprocal of the coefficient of fluctuation of speed is known as coefficient of steadiness and is denoted by m,

$$m = \frac{1}{C_S} = \frac{N}{N_1 - N_2}$$

**Energy Stored in a Flywheel:**

When a flywheel absorbs energy, its speed increases and when it gives up energy, its speed decreases.

Let $m$ = Mass of the flywheel in kg,

$k$ = Radius of gyration of the flywheel in metres,

I = Mass moment of inertia of the flywheel about its axis of rotation in kg-$m^2$ = $m.k^2$,

$N_1$ and $N_2$ = Maximum and minimum speeds during the cycle in r.p.m.,

$\omega_1$ and $\omega_2$ = Maximum and minimum angular speeds during the cycle in rad/s,

$N$ = Mean speed during the cycle in r.p.m. $= \frac{N_1 + N_2}{2}$,

$\omega$ = Mean angular speed during the cycle in rad/s $= \frac{\omega_1 + \omega_2}{2}$,

$C_S$ = Coefficient of fluctuation of speed, $= \frac{N_1 - N_2}{N}$ or $\frac{\omega_1 - \omega_2}{\omega}$

We know that the mean kinetic energy of the flywheel,

$$E = \frac{1}{2} \times I.\omega^2 = \frac{1}{2} \times m.k^2.\omega^2 \quad \text{(in N-m or joules)}$$

As the speed of the flywheel changes from $\omega_1$ to $\omega_2$, the maximum fluctuation of energy,

$$\Delta E = \text{Maximum K.E.} - \text{Minimum K.E.}$$

$$= \frac{1}{2} \times I(\omega_1)^2 - \frac{1}{2} \times I(\omega_2)^2 = \frac{1}{2} \times I\left[(\omega_1)^2 - (\omega_2)^2\right]$$

$$= \frac{1}{2} \times I(\omega_1 + \omega_2)(\omega_1 - \omega_2) = I.\omega(\omega_1 - \omega_2) \qquad ...(i)$$

$$...\left(\because \omega = \frac{\omega_1 + \omega_2}{2}\right)$$

$$= I.\omega^2\left(\frac{\omega_1 - \omega_2}{\omega}\right) \qquad \text{... (Multiplying and dividing by } \omega)$$

$$= I.\omega^2.C_S = m.k^2.\omega^2.C_S \qquad ...(\because I = m.k^2) \quad ...(ii)$$

$$= 2.E.C_S \text{ (in N–m or joules)} \qquad ...\left(\because E = \frac{1}{2} \times I.\omega^2\right) ...(iii)$$

The radius of gyration ($k$) may be taken equal to the mean radius of the rim ($R$), because the thickness of rim is very small as compared to the diameter of rim. Therefore, substituting $k = R$, in equation (*ii*), we have

$$\Delta E = m.R^2.\omega^2.C_S = m.v^2.C_S$$

where $v$ = Mean linear velocity (*i.e.* at the mean radius) in m/s = $\omega.R$

**Notes. 1.** Since $\omega = 2\pi N/60$, therefore equation (*i*) may be written as

$$\Delta E = I \times \frac{2\pi N}{60}\left(\frac{2\pi N_1}{60} - \frac{2\pi N_2}{60}\right) = \frac{4\pi^2}{3600} \times I \times N(N_1 - N_2)$$

$$= \frac{\pi^2}{900} \times m.k^2.N(N_1 - N_2)$$

$$= \frac{\pi^2}{900} \times m.k^2.N^2.C_S \qquad ...\left(\because C_S = \frac{N_1 - N_2}{N}\right)$$

2. In the above expressions, only the mass moment of inertia of the flywheel rim (I) is considered and the mass moment of inertia of the hub and arms is neglected. This is due to the fact that the major portion of the mass of the flywheel is in the rim and a small portion is in the hub and arms. Also the hub and arms are nearer to the axis of rotation, therefore the mass moment of inertia of the hub and arms is small.

**Example 16.1.** *The mass of flywheel of an engine is 6.5 tonnes and the radius of gyration is 1.8 metres. It is found from the turning moment diagram that the fluctuation of energy is 56 kN-m. If the mean speed of the engine is 120 r.p.m., find the maximum and minimum speeds.*

**Solution.** Given : $m$ = 6.5 t = 6500 kg ; $k$ = 1.8 m ; $\Delta E$ = 56 kN-m = 56 × 10³ N-m ; $N$ = 120 r.p.m.

Let $N_1$ and $N_2$ = Maximum and minimum speeds respectively.

We know that fluctuation of energy ($\Delta E$),

$$56 \times 10^3 = \frac{\pi^2}{900} \times m.k^2.N(N_1 - N_2) = \frac{\pi^2}{900} \times 6500\,(1.8)^2\,120\,(N_1 - N_2)$$

$$= 27\,715\,(N_1 - N_2)$$

$$\therefore \quad N_1 - N_2 = 56 \times 10^3 / 27\,715 = 2 \text{ r.p.m.} \qquad ...(i)$$

We also know that mean speed ($N$),

$$120 = \frac{N_1 + N_2}{2} \text{ or } N_1 + N_2 = 120 \times 2 = 240 \text{ r.p.m.} \qquad ...(ii)$$

From equations (*i*) and (*ii*),

$N_1$ = 121 r.p.m., and $N_2$ = 119 r.p.m. **Ans.**

**Example 16.2.** *The flywheel of a steam engine has a radius of gyration of 1 m and mass 2500 kg. The starting torque of the steam engine is 1500 N-m and may be assumed constant. Determine: 1. the angular acceleration of the flywheel, and 2. the kinetic energy of the flywheel after 10 seconds from the start.*

**Solution.** Given : $k = 1$ m ; $m = 2500$ kg ; $T = 1500$ N-m

**1.** ***Angular acceleration of the flywheel***

Let $\alpha$ = Angular acceleration of the flywheel.

We know that mass moment of inertia of the flywheel,

$$I = m.k^2 = 2500 \times 1^2 = 2500 \text{ kg-m}^2$$

∴ Starting torque of the engine ($T$),

$$1500 = I.\alpha = 2500 \times \alpha \quad \text{or} \quad \alpha = 1500 / 2500 = 0.6 \text{ rad/s}^2 \quad \textbf{Ans.}$$

**2.** ***Kinetic energy of the flywheel***

First of all, let us find out the angular speed of the flywheel after 10 seconds from the start (*i.e.* from rest), assuming uniform acceleration.

Let $\omega_1$ = Angular speed at rest = 0

$\omega_2$ = Angular speed after 10 seconds, and

$t$ = Time in seconds.

We know that $\omega_2 = \omega_1 + \alpha t = 0 + 0.6 \times 10 = 6$ rad/s

∴ Kinetic energy of the flywheel

$$= \frac{1}{2} \times I (\omega_2)^2 = \frac{1}{2} \times 2500 \times 6^2 = 45\ 000 \text{ N-m} = 45 \text{ kN-m} \quad \textbf{Ans.}$$



**Fluctuation of Energy:**

- The fluctuation of energy may be determined by the turning moment diagram for one complete cycle rotation.
- The vibration of energy above and below the mean resisting torque line in the turning moment diagram are called fluctuation of energy.
- The difference between the maximum and the minimum energy is known as maximum fluctuation of energy

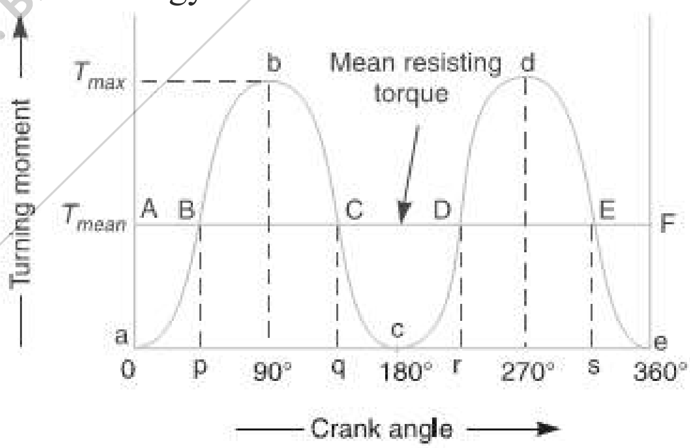



**Fig. 16.1.** Turning moment diagram for a single cylinder, double acting steam engine.

The fluctuation of energy may be determined by the turning moment diagram for one complete cycle of operation. Consider the turning moment diagram for a single cylinder double acting steam engine as shown in Fig. 16.1. We see that the mean resisting torque line AF cuts the turning moment diagram at points B, C, D and E. When the crank moves from a to p, the work done by the engine is equal to the area *aBp*, whereas the energy required is represented by the area *aABp*. In other words, the engine has done less work (equal to the area a AB) than the requirement. This amount of energy is taken from the flywheel and hence the speed of the flywheel decreases. Now the crank moves from p to q, the work done by the engine is equal to the area *pBbCq*, whereas the requirement of energy is represented by the area *pBCq*. Therefore, the engine has done more work than the requirement. This excess work (equal to the area *BbC*) is stored in the flywheel and hence the speed of the flywheel increases while the crank moves from p to q.

Similarly, when the crank moves from q to r, more work is taken from the engine than is developed. This loss of work is represented by the area *CcD*. To supply this loss, the flywheel gives up some of its energy and thus the speed decreases while the crank moves from q to r. As the crank moves from r to s, excess energy is again developed given by the area *DdE* and the speed again increases. As the piston moves from s to e, again there is a loss of work and the speed decreases. The variations of energy above and below the mean resisting torque line are called fluctuations of energy. The areas *BbC*, *CcD*, *DdE*, etc. represent fluctuations of energy.

**Maximum Fluctuation of Energy:**

A little consideration will show that the engine has a maximum speed either at q or at s. This is due to the fact that the flywheel absorbs energy while the crank moves from p to q and from r to s. On the other hand, the engine has a minimum speed either at p or at r. The reason is that the flywheel gives out some of its energy when the crank moves from a to p and q to r. The difference between the maximum and the minimum energies is known as *maximum fluctuation of energy*.

A turning moment diagram for a multi-cylinder engine is shown by a wavy curve in Fig. 16.4. The horizontal line *AG* represents the mean torque line. Let $a_1$, $a_3$, $a_5$ be the areas above the mean torque line and $a_2$, $a_4$ and $a_6$ be the areas below the mean torque line. These areas represent some quantity of energy which is either added or subtracted from the energy of the moving parts of the engine.

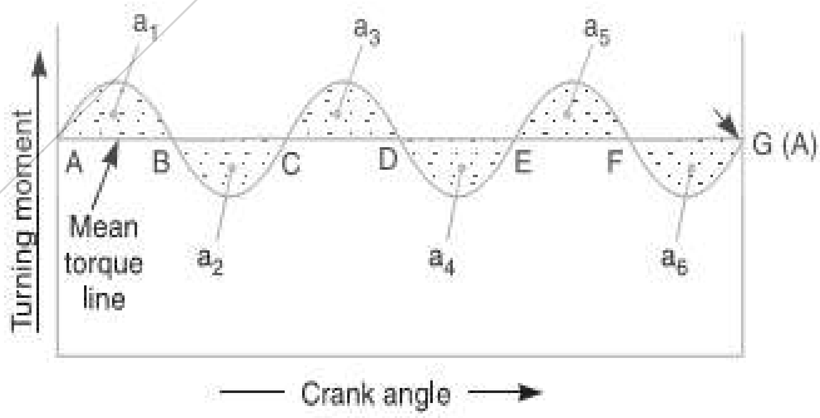



**Fig. 16.4. Determination of maximum fluctuation of energy.**

Let the energy in the flywheel at $A = E$, then from Fig. 16.4, we have

Energy at $B = E + a_1$

Energy at $C = E + a_1 - a_2$

Energy at $D = E + a_1 - a_2 + a_3$

Energy at $E = E + a_1 - a_2 + a_3 - a_4$

Energy at $F = E + a_1 - a_2 + a_3 - a_4 + a_5$

Energy at $G = E + a_1 - a_2 + a_3 - a_4 + a_5 - a_6$

= Energy at $A$ (*i.e.* cycle repeats after $G$)

A flywheel stores energy when the supply is in excess and releases energy when energy is in deficit.

Let us now suppose that the greatest of these energies is at $B$ and least at $E$. Therefore,

Maximum energy in flywheel

$$= E + a_1$$

Minimum energy in the flywheel

$$= E + a_1 - a_2 + a_3 - a_4$$

∴ Maximum fluctuation of energy,

$$\Delta E = \text{Maximum energy} - \text{Minimum energy}$$

$$= (E + a_1) - (E + a_1 - a_2 + a_3 - a_4) = a_2 - a_3 + a_4$$

## Coefficient of Fluctuation of Energy:

It may be defined as the **ratio of the maximum fluctuation of energy to the work done per cycle.** Mathematically, coefficient of fluctuation of energy,

$$C_E = \frac{\text{Maximum fluctuation of energy}}{\text{Work done per cycle}}$$

The work done per cycle (in N-m or joules) may be obtained by using the following two relations :

1. Work done per cycle $= T_{mean} \times \theta$

where $T_{mean}$ = Mean torque, and

$\theta$ = Angle turned (in radians), in one revolution.

= $2\pi$, in case of steam engine and two stroke internal combustion engines

= $4\pi$, in case of four stroke internal combustion engines.

The mean torque ($T_{mean}$) in N-m may be obtained by using the following relation :

$$T_{mean} = \frac{P \times 60}{2\pi N} = \frac{P}{\omega}$$

where $P$ = Power transmitted in watts,

$N$ = Speed in r.p.m., and

$\omega$ = Angular speed in rad/s $= 2\pi N/60$

2. The work done per cycle may also be obtained by using the following relation :

$$\text{Work done per cycle} = \frac{P \times 60}{n}$$

where $n$ = Number of working strokes per minute,

= $N$, in case of steam engines and two stroke internal combustion engines,

= $N/2$, in case of four stroke internal combustion engines.

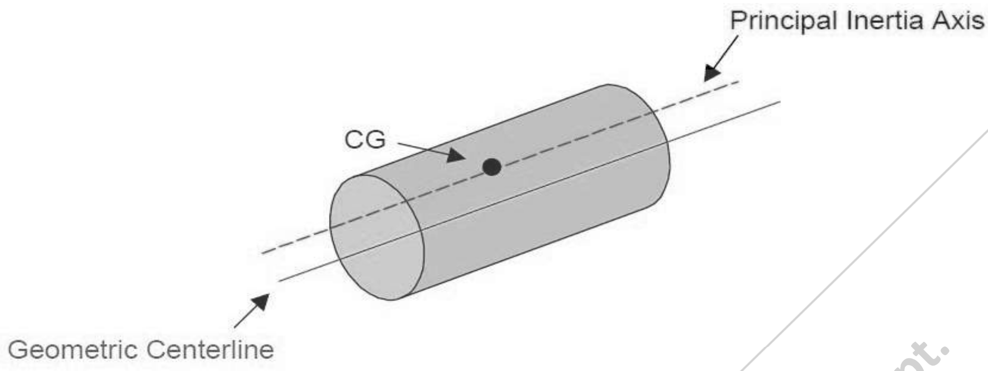
Table 16.1. Coefficient of fluctuation of energy ($C_E$) for steam and internal combustion engines.

| *S.No.* | *Type of engine* | *Coefficient of fluctuation of energy ($C_E$)* |
|---|---|---|
| 1. | Single cylinder, double acting steam engine | 0.21 |
| 2. | Cross-compound steam engine | 0.096 |
| 3. | Single cylinder, single acting, four stroke gas engine | 1.93 |
| 4. | Four cylinders, single acting, four stroke gas engine | 0.066 |
| 5. | Six cylinders, single acting, four stroke gas engine | 0.031 |

# Chapter-05: Balancing of Machine

**Unbalance**

The condition which exists in a rotor when vibratory force or motion is imparted to its bearings as a result of centrifugal forces is called *unbalance* or the *uneven distribution of mass* about a rotor's rotating centerline.




**Rotating Centerline**

The rotating centerline being defined as the axis about which the rotor would rotate if not constrained by its bearings. (Also called the Principle Inertia Axis or PIA).

**Geometric Centerline**

The geometric centerline being the physical centerline of the rotor.

When the two centerlines are coincident, then the rotor will be in a state of balance. When they are apart, the rotor will be unbalanced.

**Causes of Unbalance**

In the design of rotating parts of a machine every care is taken to eliminate any out of balance or couple, but there will be always some residual unbalance left in the finished part because of

1. Slight variation in the density of the material or
2. Inaccuracies in the casting or
3. Inaccuracies in machining of the parts.

**Why Balancing ?**

1) A level of unbalance that is acceptable at a low speed is completely unacceptable at a higher speed.
2) As machines get bigger and go faster, the effect of the unbalance is much more severe.
3) The force caused by unbalance increases by the square of the speed.
4) If the speed is doubled, the force quadruples; if the speed is tripled the force increases by a factor of nine.
5) Identifying and correcting the mass distribution and thus minimizing the force and resultant vibration is very very important

**Balancing**

Balancing is the technique of correcting or eliminating unwanted inertia forces or moments in rotating or reciprocating masses and is achieved by changing the location of the mass centers.

The objectives of balancing an engine are to ensure:

- That the centre of gravity of the system remains stationery during a complete revolution of the crank shaft and
- That the couples involved in acceleration of the different moving parts balance each other.

**Balancing of Rotating Masses**

Whenever a certain mass is attached to a rotating shaft, it exerts some centrifugal force, whose effect is to bend the shaft and to produce vibrations in it. In order to prevent the effect of centrifugal force, another mass is attached to the opposite side of the shaft, at such a position so as to balance the effect of the centrifugal force of the first mass. This is done in such a way that the centrifugal forces of both the masses are made to be equal and opposite. *The process of providing the second mass in order to counteract the effect of the centrifugal force of the first mass is called balancing of rotating masses*.

The following cases are important from the subject point of view:

***1. Balancing of a Single Rotating Mass By a Single Mass Rotating in the Same Plane:***

Consider a disturbing mass $m_1$ attached to a shaft rotating at ω rad/s as shown in Fig. 21.1. Let $r_1$ be the radius of rotation of the mass $m_1$ (i.e. distance between the axis of rotation of the shaft and the centre of gravity of the mass $m_1$).

We know that the centrifugal force exerted by the mass $m_1$ on the shaft,

$$F_{C1} = m_1 \cdot \omega^2 \cdot r_1 \qquad \ldots (i)$$

This centrifugal force acts radially outwards and thus produces bending moment on the shaft. In order to counteract the effect of this force, a balancing mass ($m_2$) may be attached in the same plane of rotation as that of disturbing mass ($m_1$) such that the centrifugal forces due to the two masses are equal and opposite.

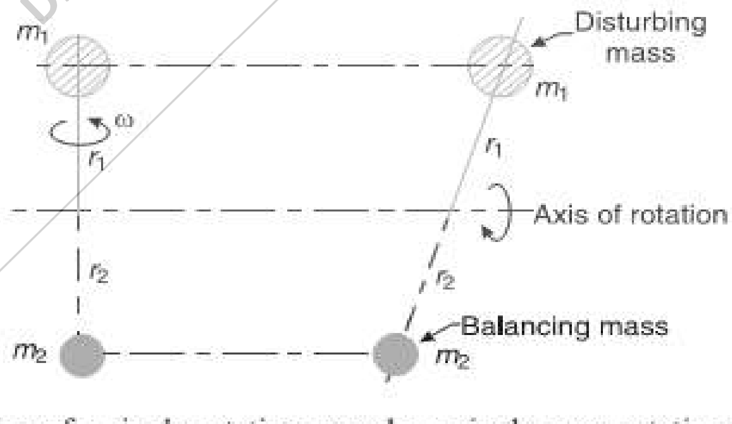



**Fig. 21.1.** Balancing of a single rotating mass by a single mass rotating in the same plane.

Let $r_2$ = Radius of rotation of the balancing mass $m_2$ (i.e. distance between the axis of rotation of the shaft and the centre of gravity of mass $m_2$).

∴ Centrifugal force due to mass $m_2$,

$$F_{C2} = m_2 \cdot \omega^2 \cdot r_2 \quad \ldots (ii)$$

Equating equations (*i*) and (*ii*),

$$m_1.\omega^2 \cdot r_1 = m_2 \cdot \omega^2 \cdot r_2 \quad \text{or} \quad m_1 \cdot r_1 = m_2 \cdot r_2$$

***2. Balancing of a Single Rotating Mass By Two Masses Rotating in Different Planes:***

The previous type of arrangement for balancing gives rise to a couple which tends to rock the shaft in its bearings. Therefore in order to put the system in complete balance, two balancing masses are placed in two different planes, parallel to the plane of rotation of the disturbing mass, in such a way that they satisfy the following two conditions of equilibrium.

***i.)*** The net dynamic force acting on the shaft is equal to zero. This requires that the line of action of three centrifugal forces must be the same. In other words, the centre of the masses of the system must lie on the axis of rotation. This is the condition for static balancing.

***ii.)*** The net couple due to the dynamic forces acting on the shaft is equal to zero. In other words, the algebraic sum of the moments about any point in the plane must be zero.

The conditions *(1)* and *(2)* together give dynamic balancing. The following two possibilities may arise while attaching the two balancing masses:

***i.)*** The plane of the disturbing mass may be in between the planes of the two balancing masses, and

***ii.)*** The plane of the disturbing mass may lie on the left or right of the two planes containing the balancing masses.

We shall now discuss both the above cases.

**Case-I**

Consider a disturbing mass m lying in a plane A to be balanced by two rotating masses $m_1$and $m_2$ lying in two different planes L and M as shown in Fig. 21.2. Let r, $r_1$ and $r_2$ be the radii of rotation of the masses in planes A, L and M respectively.

Let $l_1$ = Distance between the planes A and L,

$l_2$ = Distance between the planes A and M, and

$l$ = Distance between the planes L and M.

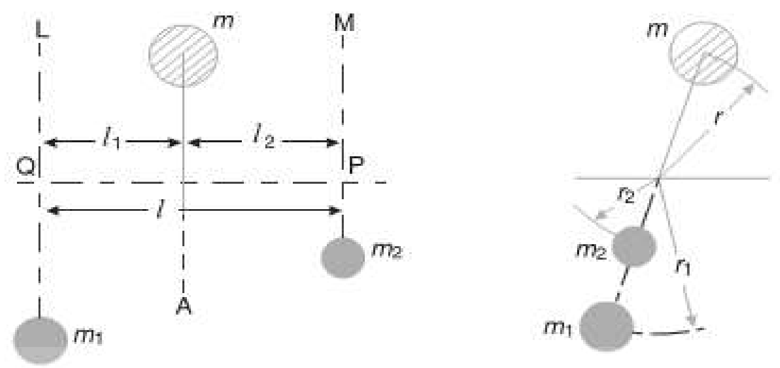



**Fig. 21.2.** Balancing of a single rotating mass by two rotating masses in different planes when the plane of single rotating mass lies in between the planes of two balancing masses.

We know that the centrifugal force exerted by the mass $m$ in the plane $A$,

$$F_C = m \cdot \omega^2 \cdot r$$

Similarly, the centrifugal force exerted by the mass $m_1$ in the plane $L$,

$$F_{C1} = m_1 \cdot \omega^2 \cdot r_1$$

and, the centrifugal force exerted by the mass $m_2$ in the plane $M$,

$$F_{C2} = m_2 \cdot \omega^2 \cdot r_2$$

Since the net force acting on the shaft must be equal to zero, therefore the centrifugal force on the disturbing mass must be equal to the sum of the centrifugal forces on the balancing masses, therefore

$$F_C = F_{C1} + F_{C2} \quad \text{or} \quad m \cdot \omega^2 \cdot r = m_1 . \omega^2 \cdot r_1 + m_2 \cdot \omega^2 \cdot r_2$$

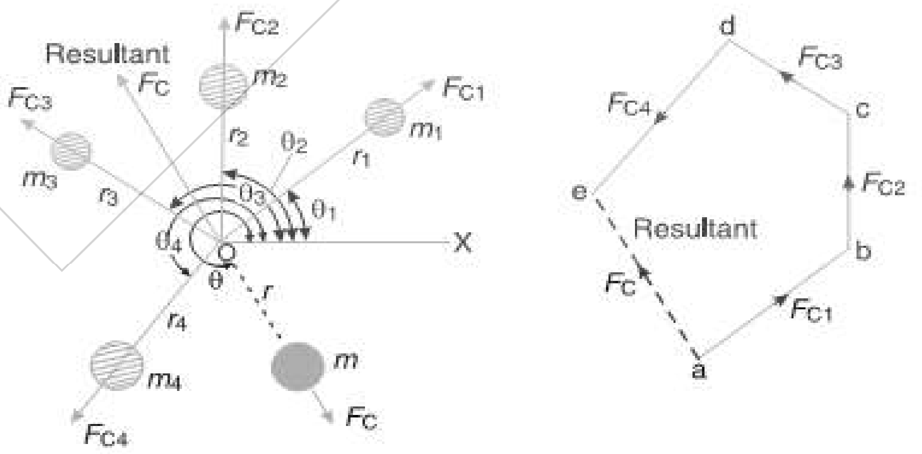
$$\therefore \quad m \cdot r = m_1 \cdot r_1 + m_2 \cdot r_2 \qquad \ldots (i)$$

Equation *(i)* represents the condition for static balance.

***3. Balancing of Several Masses Rotating in the Same Plane:***

Consider any number of masses (say four) of magnitude $m_1$, $m_2$, $m_3$ and $m_4$ at distances of $r_1$, $r_2$, $r_3$ and $r_4$ from the axis of the rotating shaft. Let $\theta_1$, $\theta_2$, $\theta_3$ and $\theta_4$, be the angles of these masses with the horizontal line OX, as shown in Fig. 21.4 (a). Let these masses rotate about an axis through O and perpendicular to the plane of paper, with a constant angular velocity of ω rad/s.

The magnitude and position of the balancing mass may be found out as below:




(*a*) Space diagram. (*b*) Vector diagram.

**Fig. 21.4.** Balancing of several masses rotating in the same plane.

The magnitude and direction of the balancing mass may be obtained, analytically, as discussed below:

**1.** First of all, find out the centrifugal force* (or the product of the mass and its radius of rotation) exerted by each mass on the rotating shaft.

**2.** Resolve the centrifugal forces horizontally and vertically and find their sums, *i.e.* $\Sigma H$ and $\Sigma V$. We know that

Sum of horizontal components of the centrifugal forces,

$$\Sigma H = m_1 \cdot r_1 \cos\theta_1 + m_2 \cdot r_2 \cos\theta_2 + \ldots\ldots$$

and sum of vertical components of the centrifugal forces,

$$\Sigma V = m_1 \cdot r_1 \sin\theta_1 + m_2 \cdot r_2 \sin\theta_2 + \ldots\ldots$$

**3.** Magnitude of the resultant centrifugal force,

$$F_C = \sqrt{(\Sigma H)^2 + (\Sigma V)^2}$$

**4.** If $\theta$ is the angle, which the resultant force makes with the horizontal, then

$$\tan\theta = \Sigma V / \Sigma H$$

**5.** The balancing force is then equal to the resultant force, but in ***opposite direction.***

**6.** Now find out the magnitude of the balancing mass, such that

$$F_C = m \cdot r$$

where $m$ = Balancing mass, and

$r$ = Its radius of rotation.

**Example 21.1.** *Four masses $m_1$, $m_2$, $m_3$ and $m_4$ are 200 kg, 300 kg, 240 kg and 260 kg respectively. The corresponding radii of rotation are 0.2 m, 0.15 m, 0.25 m and 0.3 m respectively and the angles between successive masses are 45°, 75° and 135°. Find the position and magnitude of the balance mass required, if its radius of rotation is 0.2 m.*

**Solution.** Given : $m_1$ = 200 kg ; $m_2$ = 300 kg ; $m_3$ = 240 kg ; $m_4$ = 260 kg ; $r_1$ = 0.2 m ; $r_2$ = 0.15 m ; $r_3$ = 0.25 m ; $r_4$ = 0.3 m ; $\theta_1$ = 0° ; $\theta_2$ = 45° ; $\theta_3$ = 45° + 75° = 120° ; $\theta_4$ = 45° + 75° + 135° = 255° ; $r$ = 0.2 m

Let $m$ = Balancing mass, and

$\theta$ = The angle which the balancing mass makes with $m_1$.

Since the magnitude of centrifugal forces are proportional to the product of each mass and its radius, therefore

$$m_1 \cdot r_1 = 200 \times 0.2 = 40 \text{ kg-m}$$
$$m_2 \cdot r_2 = 300 \times 0.15 = 45 \text{ kg-m}$$
$$m_3 \cdot r_3 = 240 \times 0.25 = 60 \text{ kg-m}$$
$$m_4 \cdot r_4 = 260 \times 0.3 = 78 \text{ kg-m}$$

The problem may, now, be solved either analytically or graphically. But we shall solve the problem by both the methods one by one.

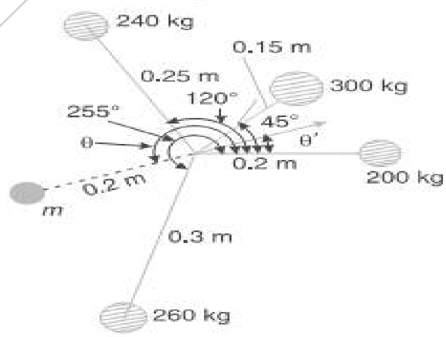



Fig. 21.5

**1.** ***Analytical method***

The space diagram is shown in Fig. 21.5.

Resolving $m_1.r_1$, $m_2.r_2$, $m_3.r_3$ and $m_4.r_4$ horizontally,

$$\Sigma H = m_1 \cdot r_1 \cos\theta_1 + m_2 \cdot r_2 \cos\theta_2 + m_3 \cdot r_3 \cos\theta_3 + m_4 \cdot r_4 \cos\theta_4$$
$$= 40\cos 0° + 45\cos 45° + 60\cos 120° + 78\cos 255°$$
$$= 40 + 31.8 - 30 - 20.2 = 21.6 \text{ kg-m}$$

Now resolving vertically,

$$\Sigma V = m_1 \cdot r_1 \sin\theta_1 + m_2 \cdot r_2 \sin\theta_2 + m_3 \cdot r_3 \sin\theta_3 + m_4 \cdot r_4 \sin\theta_4$$
$$= 40\sin 0° + 45\sin 45° + 60\sin 120° + 78\sin 255°$$
$$= 0 + 31.8 + 52 - 75.3 = 8.5 \text{ kg-m}$$

$\therefore$ Resultant, $R = \sqrt{(\Sigma H)^2 + (\Sigma V)^2} = \sqrt{(21.6)^2 + (8.5)^2} = 23.2$ kg-m

We know that

$$m \cdot r = R = 23.2 \quad \text{or} \quad m = 23.2 / r = 23.2 / 0.2 = 116 \text{ kg} \quad \textbf{Ans.}$$

and $\tan\theta' = \Sigma V / \Sigma H = 8.5 / 21.6 = 0.3935$ or $\theta' = 21.48°$

Since $\theta'$ is the angle of the resultant $R$ from the horizontal mass of 200 kg, therefore the angle of the balancing mass from the horizontal mass of 200 kg,

$$\theta = 180° + 21.48° = 201.48° \quad \textbf{Ans.}$$

### *4. Balancing of Several Masses Rotating in Different Planes:*

When several masses revolve in different planes, they may be transferred to a reference plane (briefly written as R.P.), which may be defined as the plane passing through a point on the axis of rotation and perpendicular to it. The effect of transferring a revolving mass (in one plane) to a reference plane is to cause a force of magnitude equal to the centrifugal force of the revolving mass to act in the reference plane, together with a couple of magnitude equal to the product of the force and the distance between the plane of rotation and the reference plane. In order to have a complete balance of the several revolving masses in different planes, the following two conditions must be satisfied:

1. The forces in the reference plane must balance, i.e. the resultant force must be zero.
2. The couples about the reference plane must balance, i.e. the resultant couple must be zero.

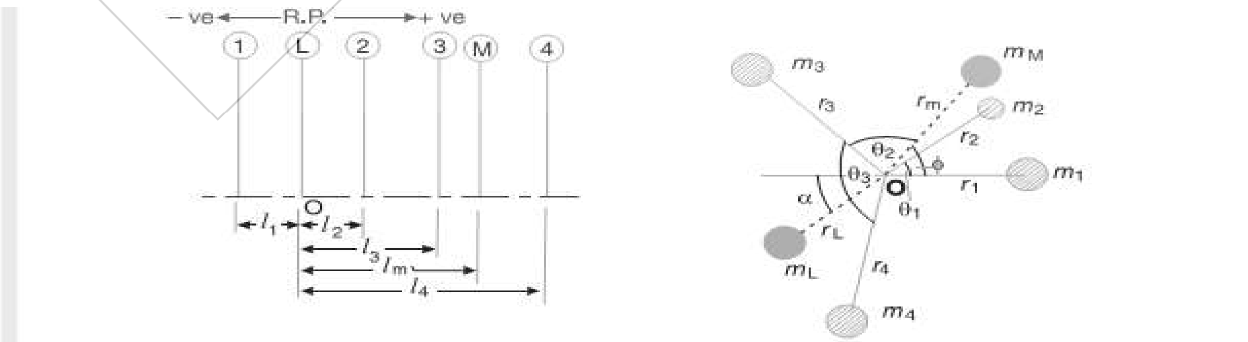
Let us now consider four masses $m_1$, $m_2$, $m_3$ and $m_4$ revolving in planes 1, 2, 3 and 4 respectively as shown in Fig. 21.7 (a). The relative angular positions of these masses are shown in the end view [Fig. 21.7 (b)]. The magnitude of the balancing masses $m_L$ and $m_M$ in planes L and M may be obtained as discussed below:

*1)* Take one of the planes, say L as the reference plane (R.P.). The distances of all the other planes to the left of the reference plane may be regarded as negative, and those to the right as positive.

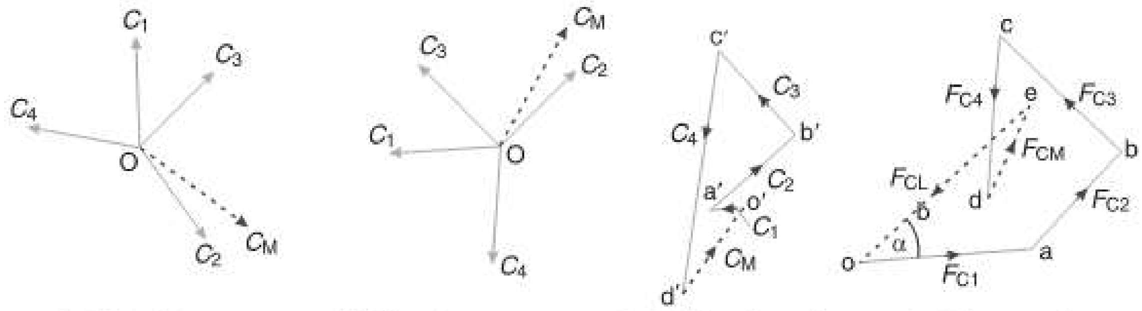
*2)* Tabulate the data as shown in Table 21.1. The planes are tabulated in the same order in which they occur, reading from left to right.

**Table 21.1**

| *Plane* (1) | *Mass (m)* (2) | *Radius(r)* (3) | *Cent.force* $\div \omega^2$ *(m.r)* (4) | *Distance from Plane L (l)* (5) | *Couple* $\div \omega^2$ *(m.r.l)* (6) |
|---|---|---|---|---|---|
| 1 | $m_1$ | $r_1$ | $m_1.r_1$ | $-l_1$ | $-m_1.r_1.l_1$ |
| *L(R.P.)* | $m_L$ | $r_L$ | $m_L.r_L$ | 0 | 0 |
| 2 | $m_2$ | $r_2$ | $m_2.r_2$ | $l_2$ | $m_2.r_2.l_2$ |
| 3 | $m_3$ | $r_3$ | $m_3.r_3$ | $l_3$ | $m_3.r_3.l_3$ |
| *M* | $m_M$ | $r_M$ | $m_M.r_M$ | $l_M$ | $m_M.r_M.l_M$ |
| 4 | $m_4$ | $r_4$ | $m_4.r_4$ | $l_4$ | $m_4.r_4.l_4$ |




(*a*) Position of planes of the masses. (*b*) Angular position of the masses.




(*c*) Couple vector. (*d*) Couple vectors turned counter clockwise through a right angle. (*e*) Couple polygon. (*f*) Force polygon.

**Fig. 21.7. Balancing of several masses rotating in different planes.**

*3)* A couple may be represented by a vector drawn perpendicular to the plane of the couple. The couple $C_1$ introduced by transferring $m_1$ to the reference plane through O is proportional to $m_1.r_1.l_1$ and acts in a plane through $Om_1$ and perpendicular to the paper. The vector representing this couple is drawn in the plane of the paper and perpendicular to $Om_1$ as shown by $OC_1$ in Fig. 21.7 (c). Similarly, the vectors $OC_2$, $OC_3$ and $OC_4$ are drawn perpendicular to $Om_2$, $Om_3$ and $Om_4$ respectively and in the plane of the paper.

*4)* The couple vectors as discussed above, are turned counter clockwise through a right angle for convenience of drawing as shown in Fig. 21.7 (d). We see that their relative position remains unaffected. Now the vectors $OC_2$, $OC_3$ and $OC_4$ are parallel and in the same direction as $Om_2$, $Om_3$ and $Om_4$, while the vector $OC_1$ is parallel to $Om_1$ but in *opposite direction. Hence the couple vectors are drawn radially outwards for the masses on one side of the reference plane and radially inward for the masses on the other side of the reference plane.

*5)* Now draw the couple polygon as shown in Fig. 21.7 (e). The vector $d'o'$ represents the balanced couple. Since the balanced couple $C_M$ is proportional to $m_M.r_M.l_M$, therefore

$$C_M = m_M \cdot r_M \cdot l_M = \text{vector } d'o' \quad \text{or} \quad m_M = \frac{\text{vector } d'o'}{r_M \cdot l_M}$$

From this expression, the value of the balancing mass $m_M$ in the plane M may be obtained, and the angle of inclination $\varphi$ of this mass may be measured from Fig. 21.7 (b).

*6)* Now draw the force polygon as shown in Fig. 21.7 (f). The vector eo (in the direction from e to o) represents the balanced force. Since the balanced force is proportional to $m_L.r_L$, therefore,

$$m_L \cdot r_L = \text{vector } eo \quad \text{or} \quad m_L = \frac{\text{vector } eo}{r_L}$$

From this expression, the value of the balancing mass $m_L$ in the plane L may be obtained and the angle of inclination α of this mass with the horizontal may be measured from Fig. 21.7 (b).

**Balancing of Reciprocating Parts**

The resultant of all the forces acting on the body of the engine due to inertia forces only is known as *unbalanced force* or *shaking force*. Consider a horizontal reciprocating engine mechanism as shown in Fig. 22.1.

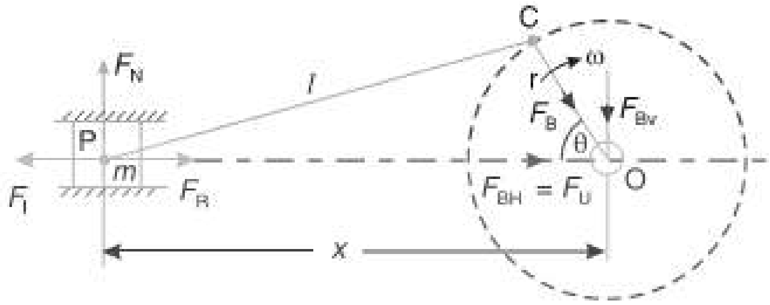



**Fig. 22.1.** Reciprocating engine mechanism.

Let $F_R$ = Force required to accelerate the reciprocating parts,

$F_I$ = Inertia force due to reciprocating parts,

$F_N$ = Force on the sides of the cylinder walls or normal force acting on the cross-head guides, and

$F_B$ = Force acting on the crankshaft bearing or main bearing.

Since $F_R$ and $F_I$ are equal in magnitude but opposite in direction, therefore they balance each other. The horizontal component of $F_B$ (i.e. $F_{BH}$) acting along the line of reciprocation is also equal and opposite to $F_I$. This force $F_{BH} = F_U$ is an unbalanced force or shaking force and required to be properly balanced.

The force on the sides of the cylinder walls ($F_N$) and the vertical component of $F_B$ (i.e. $F_{BV}$) are equal and opposite and thus form a shaking couple of magnitude $F_N \times x$ *or* $F_{BV} \times x$.

Thus the purpose of balancing the reciprocating masses is to eliminate the shaking force and a shaking couple. In most of the mechanisms, we can reduce the shaking force and a shaking couple by adding appropriate balancing mass, but it is usually not practical to eliminate them completely. In other words, the reciprocating masses are only partially balanced.

***Note:*** The masses rotating with the crankshaft are normally balanced and they do not transmit any unbalanced or shaking force on the body of the engine.

**Primary and Secondary Unbalanced Forces of Reciprocating Masses**

Consider a reciprocating engine mechanism as shown in Fig. 22.1.

Let m = Mass of the reciprocating parts,

$l$ = Length of the connecting rod PC,

r = Radius of the crank OC,

θ = Angle of inclination of the crank with the line of stroke PO,

ω = Angular speed of the crank,

n = Ratio of length of the connecting rod to the crank radius = $l$ / r.

We have the acceleration of the reciprocating parts is approximately given by the expression.

$$a_R = \omega^2 \cdot r\left(\cos\theta + \frac{\cos 2\theta}{n}\right)$$

∴ Inertia force due to reciprocating parts or force required to accelerate the reciprocating parts,

$$F_I = F_R = \text{Mass} \times \text{acceleration} = m \cdot \omega^2 \cdot r\left(\cos\theta + \frac{\cos 2\theta}{n}\right)$$

We have the horizontal component of the force exerted on the crank shaft bearing (i.e. $F_{BH}$) is equal and opposite to inertia force ($F_I$). This force is an unbalanced one and is denoted by $F_U$.

∴ Unbalanced force,

$$F_U = m \cdot \omega^2 \cdot r\left(\cos\theta + \frac{\cos 2\theta}{n}\right) = m.\omega^2 \cdot r\cos\theta + m \cdot \omega^2 \cdot r \times \frac{\cos 2\theta}{n} = F_P + F_S$$

The expression $(m \cdot \omega^2 \cdot r\cos\theta)$ is known as ***primary unbalanced force*** and $\left(m \cdot \omega^2 \cdot r \times \frac{\cos 2\theta}{n}\right)$ is called ***secondary unbalanced force***.

**Notes:** *1.* The primary unbalanced force is maximum, when θ = 0° or 180°. Thus, the primary force is maximum twice in one revolution of the crank. The maximum primary unbalanced force is given by,

$$F_{P(max)} = m \cdot \omega^2 \cdot r$$

*2.* The secondary unbalanced force is maximum, when θ = 0°, 90°, 180° and 360°. Thus, the secondary force is maximum four times in one revolution of the crank. The maximum secondary unbalanced force is given by,

$$F_{S(max)} = m \cdot \omega^2 \times \frac{r}{n}$$

*3.* From above we see that secondary unbalanced force is 1/n times the maximum primary unbalanced force.

*4.* In case of moderate speeds, the secondary unbalanced force is so small that it may be neglected as compared to primary unbalanced force.

*5.* The unbalanced force due to reciprocating masses varies in magnitude but constant in direction while due to the revolving masses; the unbalanced force is constant in magnitude but varies in direction.

**Principles of Balancing of Reciprocating Parts**

For complete balancing of the reciprocating parts, the following condition must be fulfilled:

***a)*** Primary forces must balance, i.e. primary force polygon is enclosed.
***b)*** Primary couples must balance, i.e. primary couple polygon is enclosed.
***c)*** Secondary forces must balance, i.e. secondary forces polygon is enclosed.
***d)*** Secondary couples must balance, i.e. secondary couple polygon is enclosed.

Usually, it is not possible to satisfy all the above conditions fully for a multi-cylinder engine. Mostly some unbalanced force or couple would exist in the reciprocating engines.

**Difference between static and dynamic balancing**

| Static balancing | Dynamic balancing |
|---|---|
| It would refer to balancing in a single plane | It would refer to balancing in more than one plane |
| It is also known as primary balancing. It is a balance force due to action of gravity. | It is also known as secondary balancing. It is a balance due to action of inertia forces. |
| Rotation of fly wheels, grinding wheels, car wheels are treated as static balancing problems as most of their masses concentrated in or very near one plane. | Rotation of shaft of turbo-generator is a case of dynamic balancing problems |
| Static balance occurs when the centre of gravity of an object is on the axis of rotation | A rotating system of mass is in dynamic balance when the rotation doesn't produce any resultant centripetal force of couple. Here the mass axis is coincidental with the rotational axis. |

# Chapter-06: Vibration of Machine Parts

## Vibration

- When elastic bodies such as a spring, a beam and a shaft are displaced from the equilibrium position by the application of external forces, and then released, they execute a *vibratory motion*.
- This is due to the reason that, when a body is displaced, the internal forces in the form of elastic or strain energy are present in the body. At release, these forces bring the body to its original position.
- When the body reaches the equilibrium position, the whole of the elastic or strain energy is converted into kinetic energy due to which the body continues to move in the opposite direction.
- The whole of the kinetic energy is again converted into strain energy due to which the body again returns to the equilibrium position. In this way, the vibratory motion is repeated indefinitely.

## Related Terms

a) ***Time Period or Period of Vibration:*** It is the time interval after which the motion is repeated itself and is usually expressed in seconds.

b) ***Cycle:*** It is the motion completed during one time period.

c) ***Amplitude:*** It is defined as its maximum displacement of a vibrating body from its equilibrium position.

d) ***Frequency:*** It is the number of cycles described in one second. In S.I. units, the frequency is expressed in hertz (briefly written as Hz) which is equal to one cycle per second.

## Classification of Vibratory Motion

a) ***Free or Natural Vibrations:*** When no external force acts on the body, after giving it an initial displacement, then the body is said to be under free or natural vibrations. The frequency of the free vibrations is called *free or natural frequency*.

b) ***Forced Vibrations:*** When the body vibrates under the influence of external force, then the body is said to be under forced vibrations. The external force applied to the body is a periodic disturbing force created by unbalance. The vibrations have the same frequency as the applied force.

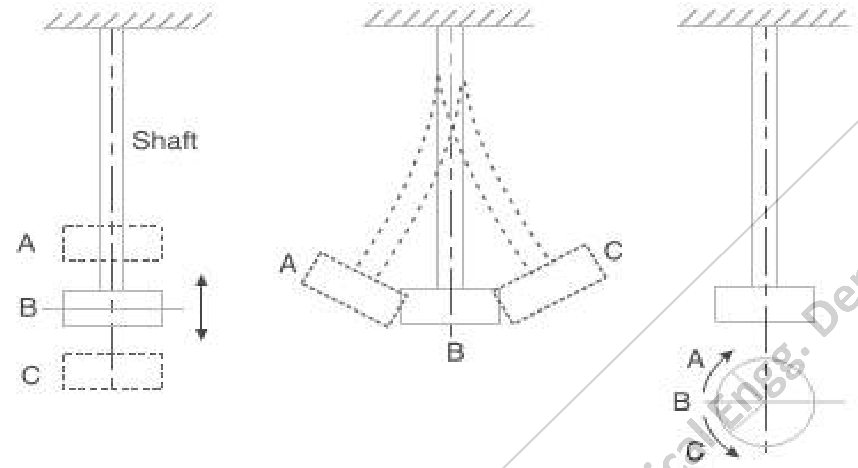
Note: When the frequency of the external force is same as that of the natural vibrations, *resonance* takes place.

c) ***Damped Vibrations:*** When there is a reduction in amplitude over every cycle of vibration, the motion is said to be damped vibration. This is due to the fact that a certain amount of energy possessed by the vibrating system is always dissipated in overcoming frictional resistances to the motion.

### Types of Free Vibrations

Consider a weightless constraint (spring or shaft) whose one end is fixed and the other end carrying a heavy disc, as shown in Fig. 23.1.

a) ***Longitudinal Vibrations:*** When the particles of the shaft or disc moves parallel to the axis of the shaft, as shown in Fig. 23.1 (a), then the vibrations are known as longitudinal vibrations. In this case, the shaft is elongated and shortened alternately and thus the tensile and compressive stresses are induced alternately in the shaft.




*B* = Mean position ; *A* and *C* = Extreme positions.

(*a*) Longitudinal vibrations. (*b*) Transverse vibrations. (*c*) Torsional vibrations.

**Fig. 23.1.** Types of free vibrations.

b) ***Transverse Vibrations:*** When the particles of the shaft or disc move approximately perpendicular to the axis of the shaft, as shown in Fig. 23.1 (b), then the vibrations are known as transverse vibrations. In this case, the shaft is straight and bent alternately and bending stresses are induced in the shaft.

c) ***Transverse Vibrations:*** When the particles of the shaft or disc move in a circle about the axis of the shaft, as shown in Fig. 23.1 (c), then the vibrations are known as torsional vibrations. In this case, the shaft is twisted and untwisted alternately and the torsional shear stresses are induced in the shaft.

Note: Consider if the limit of proportionality (i.e. stress proportional to strain) is not exceeded in the three types of vibrations, then the restoring force in longitudinal and transverse vibrations or the restoring couple in torsional vibrations which is exerted on the disc by the shaft (due to the stiffness of the shaft) is directly proportional to the displacement of the disc from its equilibrium or mean position. Hence it follows that the acceleration towards the equilibrium position is directly proportional to the displacement from that position and the vibration is, therefore, simple harmonic.

### Natural Frequency of Free Longitudinal Vibrations:

Consider a constraint (i.e. spring) of negligible mass in an unstrained position, as shown in Fig. 23.2 (a).

Let $s$ = Stiffness of the constraint. It is the force required to produce unit displacement in the direction of vibration. It is usually expressed in N/m, m = Mass of the body suspended from the constraint in kg,

W = Weight of the body in newtons = m.g,

δ = Static deflection of the spring in metres due to weight W newtons,

and x = Displacement given to the body by the external force, in metres.

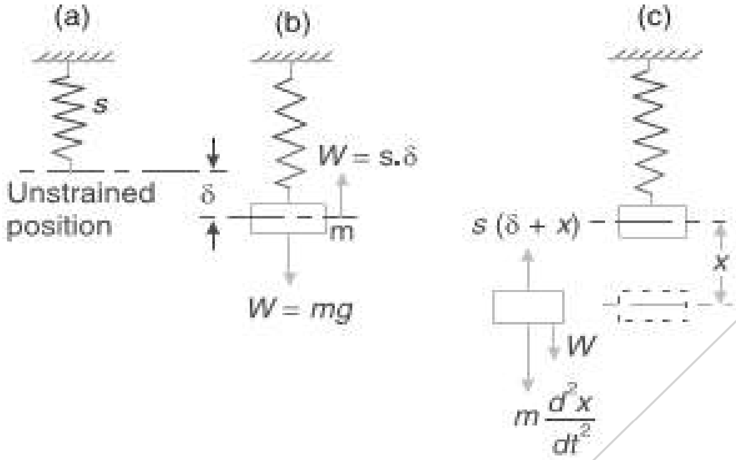



Fig. 23.2. Natural frequency of free longitudinal vibrations.

In the equilibrium position, as shown in Fig. 23.2 (b), the gravitational pull W = m.g, is balanced by a force of spring, such that W = s.δ.

Since the mass is now displaced from its equilibrium position by a distance x, as shown in Fig. 23.2 (c), and is then released, therefore after time t,

Restoring force $= W - s(\delta + x) = W - s.\delta - s.x$

$= s.\delta - s.\delta - s.x = -s.x$ . . . ($\because W = s.\delta$) . . . (*i*)

. . . (Taking upward force as negative)

and Accelerating force = Mass × Acceleration

$= m \times \frac{d^2x}{dt^2}$ . . . (Taking downward force as positive) . . . (*ii*)

Equating equations (*i*) and (*ii*), the equation of motion of the body of mass *m* after time *t* is

$$m \times \frac{d^2x}{dt^2} = -s.x \quad \text{or} \quad m \times \frac{d^2x}{dt^2} + s.x = 0$$

$$\therefore \quad \frac{d^2x}{dt^2} + \frac{s}{m} \times x = 0 \qquad \text{. . . (iii)}$$

We know that the fundamental equation of simple harmonic motion is

$$\frac{d^2x}{dt^2} + \omega^2.x = 0 \qquad \text{. . . (iv)}$$

Comparing equations (*iii*) and (*iv*), we have

$$\omega = \sqrt{\frac{s}{m}}$$

$\therefore$ Time period, $$t_p = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{s}}$$

and natural frequency, $f_n = \frac{1}{t_p} = \frac{1}{2\pi}\sqrt{\frac{s}{m}} = \frac{1}{2\pi}\sqrt{\frac{g}{\delta}}$ . . . ($\because\ m.g = s.\delta$)

Taking the value of $g$ as 9.81 m/s$^2$ and $\delta$ in metres,

$$f_n = \frac{1}{2\pi}\sqrt{\frac{9.81}{\delta}} = \frac{0.4985}{\sqrt{\delta}}\text{ Hz}$$

**Note :** The value of static deflection $\delta$ may be found out from the given conditions of the problem. For longitudinal vibrations, it may be obtained by the relation,

$$\frac{\text{Stress}}{\text{Strain}} = E \quad \text{or} \quad \frac{W}{A}\times\frac{l}{\delta} = E \quad \text{or} \quad \delta = \frac{W.l}{E.A}$$

where $\delta$ = Static deflection *i.e.* extension or compression of the constraint,

$W$ = Load attached to the free end of constraint,

$l$ = Length of the constraint,

$E$ = Young's modulus for the constraint, and

$A$ = Cross-sectional area of the constraint.

## Natural Frequency of Free Transverse Vibrations:

Consider a shaft of negligible mass, whose one end is fixed and the other end carries a body of weight $W$, as shown in Fig. 23.3.

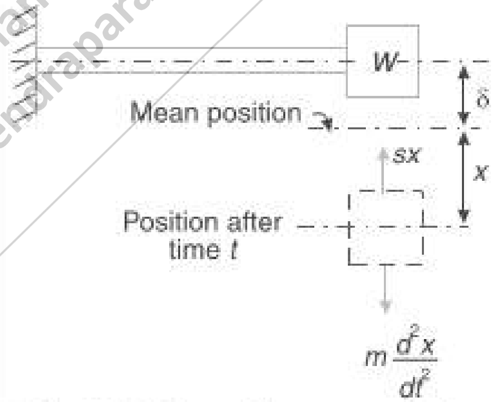


**Fig. 23.3.** Natural frequency of free transverse vibrations.

Let $s$ = Stiffness of shaft,

$\delta$ = Static deflection due to weight of the body,

$x$ = Displacement of body from mean position after time $t$.

$m$ = Mass of body = $W/g$

As discussed in the previous article,

Restoring force $= -s.x$ . . . (*i*)

and accelerating force $= m\times\frac{d^2x}{dt^2}$ . . . (*ii*)

Equating equations (*i*) and (*ii*), the equation of motion becomes

$$m\times\frac{d^2x}{dt^2} = -s.x \quad \text{or} \quad m\times\frac{d^2x}{dt^2} + s.x = 0$$

$\therefore$ $\frac{d^2x}{dt^2} + \frac{s}{m}\times x = 0$ . . . (Same as before )

Hence, the time period and the natural frequency of the transverse vibrations are same as that of longitudinal vibrations. Therefore

Time period, $t_p = 2\pi\sqrt{\frac{m}{s}}$

and natural frequency, $f_n = \frac{1}{t_p} = \frac{1}{2\pi}\sqrt{\frac{s}{m}} = \frac{1}{2\pi}\sqrt{\frac{g}{\delta}}$

**Note :** The shape of the curve, into which the vibrating shaft deflects, is identical with the static deflection curve of a cantilever beam loaded at the end. It has been proved in the text book on Strength of Materials, that the static deflection of a cantilever beam loaded at the free end is

$$\delta = \frac{Wl^3}{3EI} \text{ (in metres)}$$

where $W$ = Load at the free end, in newtons,

$l$ = Length of the shaft or beam in metres,

$E$ = Young's modulus for the material of the shaft or beam in $N/m^2$, and

$I$ = Moment of inertia of the shaft or beam in $m^4$.

**Example 23.1.** *A cantilever shaft 50 mm diameter and 300 mm long has a disc of mass 100 kg at its free end. The Young's modulus for the shaft material is 200 $GN/m^2$. Determine the frequency of longitudinal and transverse vibrations of the shaft.*

**Solution.** Given : $d$ = 50 mm = 0.05 m ; $l$ = 300 mm = 0.03 m ; $m$ = 100 kg ; $E = 200 \text{ GN/m}^2 = 200 \times 10^9 \text{ N/m}^2$

We know that cross-sectional area of the shaft,

$$A = \frac{\pi}{4} \times d^2 = \frac{\pi}{4}(0.05)^2 = 1.96 \times 10^{-3} \text{ m}^2$$

and moment of inertia of the shaft,

$$I = \frac{\pi}{64} \times d^4 = \frac{\pi}{64}(0.05)^4 = 0.3 \times 10^{-6} \text{ m}^4$$

***Frequency of longitudinal vibration***

We know that static deflection of the shaft,

$$\delta = \frac{W.l}{A.E} = \frac{100 \times 9.81 \times 0.3}{1.96 \times 10^{-3} \times 200 \times 10^9} = 0.751 \times 10^{-6} \text{ m}$$

$$\ldots(\because W = m.g)$$

$\therefore$ Frequency of longitudinal vibration,

$$f_n = \frac{0.4985}{\sqrt{\delta}} = \frac{0.4985}{\sqrt{0.751 \times 10^{-6}}} = 575 \text{ Hz} \quad \textbf{Ans.}$$

***Frequency of transverse vibration***

We know that static deflection of the shaft,

$$\delta = \frac{W.l^3}{3E.I} = \frac{100 \times 9.81 \times (0.3)^3}{3 \times 200 \times 10^9 \times 0.3 \times 10^{-6}} = 0.147 \times 10^{-3} \text{ m}$$

$\therefore$ Frequency of transverse vibration,

$$f_n = \frac{0.4985}{\sqrt{\delta}} = \frac{0.4985}{\sqrt{0.147 \times 10^{-3}}} = 41 \text{ Hz} \quad \textbf{Ans.}$$

## Natural Frequency of Free Torsional Vibrations:

Consider a shaft of negligible mass whose one end is fixed and the other end carrying a disc as shown in Fig. 24.1.

Let $\theta$ = Angular displacement of the shaft from mean position after time t in radians,

m = Mass of disc in kg,

I = Mass moment of inertia of disc in kg-m$^2$ = m.k$^2$,

k = Radius of gyration in metres,

q = Torsional stiffness of the shaft in N-m.

$\therefore$ Restoring force $= q.\theta$ ... (*i*)

and accelerating force $= I \times \frac{d^2\theta}{dt^2}$ ... (*ii*)

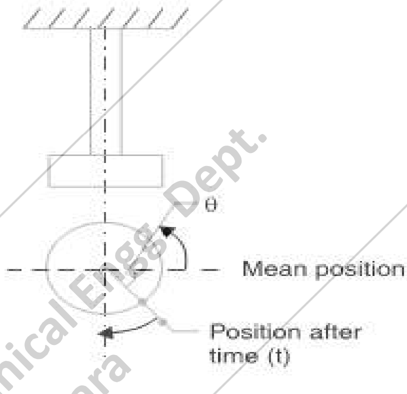


**Fig 24.1.** Natural frequency of free torsional vibrations.

Equating equations (*i*) and (*ii*), the equation of motion is

$$I \times \frac{d^2\theta}{dt^2} = -q.\theta$$

or

$$I \times \frac{d^2\theta}{dt^2} + q.\theta = 0$$

$$\therefore \quad \frac{d^2\theta}{dt^2} + \frac{q}{I} \times \theta = 0 \quad \dots (iii)$$

The fundamental equation of the simple harmonic motion is

$$\frac{d^2\theta}{dt^2} + \omega^2.x = 0 \quad \dots (iv)$$

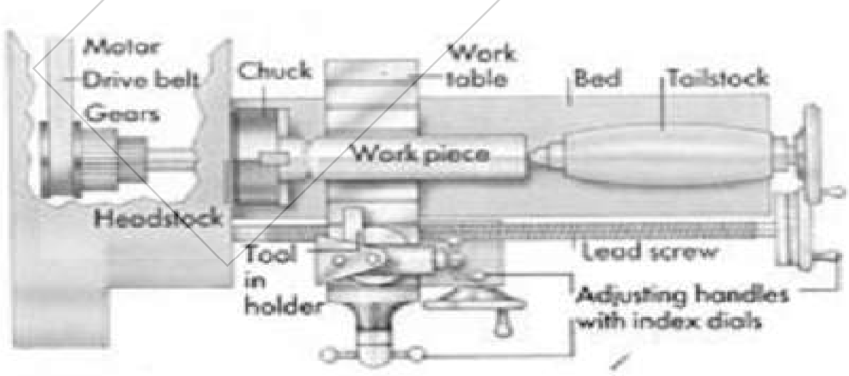
Comparing equations (*iii*) and (*iv*),

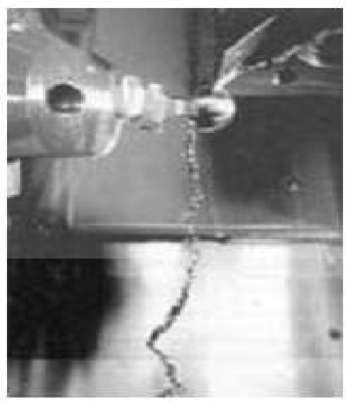
$$\omega = \sqrt{\frac{q}{I}}$$

$\therefore$ Time period, $t_p = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{I}{q}}$

and natural frequency, $f_n = \frac{1}{t_p} = \frac{1}{2\pi}\sqrt{\frac{q}{I}}$





Note : This picture is given as additional information and is not a direct example of the current chapter.

A modern lathe can create an artificial hip joint from information fed into it by a computer. Accurate drawings of the joint are first made on a computer and the information about the dimensions fed is directly into the lathe.

**Note :** The value of the torsional stiffness $q$ may be obtained from the torsion equation,

$$\frac{T}{J} = \frac{C.\theta}{l} \quad \text{or} \quad \frac{T}{\theta} = \frac{C.J}{l}$$

$$\therefore \qquad q = \frac{C.J}{l} \qquad \ldots \left(\because \frac{T}{\theta} = q\right)$$

where $C$ = Modulus of rigidity for the shaft material,

$J$ = Polar moment of inertia of the shaft cross-section,

$= \frac{\pi}{32} d^4$ ; $d$ is the diameter of the shaft, and

$l$ = Length of the shaft.

**Example 24.1.** *A shaft of 100 mm diameter and 1 metre long has one of its end fixed and the other end carries a disc of mass 500 kg at a radius of gyration of 450 mm. The modulus of rigidity for the shaft material is 80 GN/m². Determine the frequency of torsional vibrations.*

**Solution.** Given : $d$ = 100 mm = 0.1 m ; $l$ = 1 m ; $m$ = 500 kg ; $k$ = 450 mm = 0.45 m ; $C = 80$ GN/m² = $80 \times 10^9$ N/m²

We know that polar moment of inertia of the shaft,

$$J = \frac{\pi}{32} \times d^4 = \frac{\pi}{32}(0.1)^4 = 9.82 \times 10^{-6}\ \text{m}^4$$

∴ Torsional stiffness of the shaft,

$$q = \frac{C.J}{l} = \frac{80 \times 10^9 \times 9.82 \times 10^{-6}}{1} = 785.6 \times 10^3\ \text{N-m}$$

We know that mass moment of inertia of the shaft,

$$I = m.k^2 = 500(0.45)^2 = 101.25\ \text{kg-m}^2$$

∴ Frequency of torsional vibrations,

$$f_n = \frac{1}{2\pi}\sqrt{\frac{q}{I}} = \frac{1}{2\pi}\sqrt{\frac{785.6 \times 10^3}{101.25}} = \frac{88.1}{2\pi} = 14\ \text{Hz } \textbf{Ans.}$$

**Example 24.2.** *A flywheel is mounted on a vertical shaft as shown in Fig 24.2. The both ends of a shaft are fixed and its diameter is 50 mm. The flywheel has a mass of 500 kg and its radius of gyration is 0.5 m. Find the natural frequency of torsional vibrations, if the modulus of rigidity for the shaft material is 80 GN/m².*

**Solution.** Given : $d$ = 50 mm = 0.05 m ; $m$ = 500 kg ; $k$ = 0.5m; $G$ = 80 GN/m² = $84 \times 10^9$ N/m²

We know that polar moment of inertia of the shaft,

$$J = \frac{\pi}{32} \times d^4 = \frac{\pi}{32}(0.05)^4\ \text{m}^4$$

$$= 0.6 \times 10^{-6}\ \text{m}^4$$

∴ Torsional stiffness of the shaft for length $l_1$,

$$q_1 = \frac{C.J}{l_1} = \frac{84 \times 10^9 \times 0.6 \times 10^{-6}}{0.9}$$

$= 56 \times 10^3$ N-m

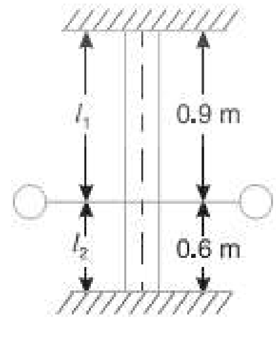


Fig 24.2

Similarly torsional stiffness of the shaft for length $l_2$,

$$q_2 = \frac{C.J}{l_2} = \frac{84\times10^9\times0.6\times10^{-6}}{0.6} = 84\times10^3 \text{ N-m}$$

∴ Total torsional stiffness of the shaft,

$$q = q_1 + q_2 = 56\times10^3 + 84\times10^3 = 140\times10^3 \text{ N-m}$$

We know that mass moment of inertia of the flywheel,

$$I = m.k^2 = 500(0.5)^2 = 125 \text{ kg-m}^2$$

∴ Natural frequency of torsional vibration,

$$f_n = \frac{1}{2\pi}\sqrt{\frac{q}{I}} = \frac{1}{2\pi}\sqrt{\frac{140\times10^3}{125}} = \frac{33.5}{2\pi} = 5.32 \text{ Hz Ans.}$$

**Causes of Vibration.**

Some of the important causes of vibration in machines are listed below:

1) Unbalanced reciprocating machine parts.
2) Unbalanced rotating machine parts.
3) Incorrect alignment of the transmission elements such as coupling etc.
4) Use of simple spur gears for power transmission.
5) Worn-out teeth of the gears of the power transmission.
6) Impact of working fluids of prime movers.
7) Loose transmission belts and chains.
8) Loose fastenings of the moving parts.
9) Vibration waves from other sources and machines installed nearly, due to improper isolations of vibrations from them.
10) Due to more material contact such as bases plates on the foundations for the pedestal bearings.
11) Non-rigid machine foundations due to lack of compact soil below, causing settlement of machine components.

**Remedies of Vibration**

Although it is impossible to eliminate the vibrations, yet these can be reduced by adopting various remedies, some of the remedies are listed below:

1) Partial balancing of reciprocating masses.
2) Balancing of unbalanced rotating masses.
3) Using helical gears instead of spur gases.
4) Proper tightening and locking of fastening and periodically ensuring it again.
5) Correcting the misalignment of rotating components and checking it from time to time.
6) Timely replacement of worn-out moving parts, slides and bearings with excessive clearance.
7) Isolating vibrations from other machines and sources by providing vibration insulation pads in the machine foundations.
8) Making machine foundations on compact ground and making them sufficiently strong, so that they do not yield or settle under the load of the machine.